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Two Essays On Economic Theory

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TWO ESSAYS ON ECONOMIC THEORY

by

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ABSTRACT

My thesis consists of two separate essays on economic theory The title of the first essay (chapter 1) is "Learning and Nash Equilibria in 3×3 Symmetric Games" and the title of the second essay (chapter 2) is "General Equilibrium Theory with Monopolistic Competition: An Introductory Analysis."

Chapter 1 explores the dynamic implications of learning models by studying fictitious play in 3×3 symmetric games. symmetric games. The basic model consists of two persons playing a symmetric normal form game with only three pure strategies repeatedly. and choosing their strategies in each period to maximize their expected payoffs in the stage game. After each play of the game each person forms his belief about his opponent's next strategy choice according to rules defined by fictitious play. It is shown that for a reasonably wide class of 3×3 symmetric games the sequence of beliefs generated by fictitious play inevitably converges to a mixed-strategy Nash equilibrium. For games that do not belong to this class. the limiting outcomes of fictitious play with identical initial beliefs are characterized. Finally. replacing fictitious play with another more sophiscated learning process yields stronger convergence results for fictitious play. These results provide useful references for future work in this area.

Chapter 2 provides an introduction to general equilibrium theory with monopolistic competition. A model of an economy with monopolistic competitive firms is constructed and studied. Each monopolistic firm perceives a demand function for its output which satisfies certain

consistency condition in equilibrium. Three basic questions in equilibrium theory are addressed in the context of the model: existence, uniqueness and pareto compatibility of monopolistic competitive equilibria. Using fairly standard assumptions, it is shown that an infinite number of monopolistic competitive equilibria exists which are supported by different systems of perceived demand functions. Fixing the perceived demand functions, however, the number of equilibria (with different prices and allocations) is generically finite. Also, different equilibria supported by different systems of perceived demand functions are in general pareto incompatible.

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This thesis is dedicated to my parents, Kwan Tong chan and Sau Tuen Sze. They had been very patient and supportive throughout the course of my study.

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CHAPTER 1

LEARNING AND NASH EQUILIBRIUM IN 3×3 SYMMETRIC GAMES

1.1 Introduction

The notion of a Nash equilibrium, though extremely useful, leaves unexplained the important point of how an equilibrium might arise. This unsatisfactory aspect of the equilibrium concept has prompted many game theorists and economists to look for a convincing explanation of the source of equilibrium. Accordingly, research on the foundation of Nash equilibrium has evolved along two different lines. The first line of research, which I shall refer to as the common knowledge approach to Nash equilibrium, tries to argue that, by making extensive assumptions regarding the rationality and the knowledge of players, players naturally come to play an equilibrium by deductive reasoning alone. Representative work of this line includes Aumann and Brandenburger (1991) and Brandenburger (1992).

The second line of research, which in recent years has attracted a lot of attention from economic theorists, strives to provide a learning foundation for Nash equilibrium by showing that under appropriate conditions, players will learn to play a Nash equilibrium if the same game is played repeatedly. Unlike the common knowledge approach, players are not assumed to begin with detailed information of their opponents' beliefs and behaviour, but instead they are given the opportunity to acquire such information through repeated interaction. In most learning models (see for example, Jordan (1991), Kalai and Lehrer (1993) and krishna (1992)), players are informed of the actions or strategy choices of their opponents after each play of the game.[1] A specified learning rule or mechanism then transforms the players' information into beliefs about their opponents' current behaviour.

Based on their beliefs, the players choose their current strategies for the stage game. This process continues as the game is repeated over and over again, and the players will continue to adjust their strategies in accordance with their revised beliefs.

Nash equilibrium in this repeated game context is viewed as the stable outcome of an adaptive dynamic process that describes the adjustment of strategic choices over time. Because of their relatively simple assumptions and dynamic nature, learning models not only provide a plausible alternative justification for equilibrium analysis but also allow the study of related issues such as the discrimination of different equilibria by their relative stability and the limiting behaviour of beliefs and strategic choices when learning does not lead to an equilibrium.

Chapter 1 of my thesis explores the implications of learning models by studying a simple learning process known as fictitious play in 3×3 symmetric games. The objective here is to gain a deeper understanding of fictitious play by focusing on simple and tractable games. I investigate the asymptotic behaviour of beliefs generated by fictitious play, with particular emphasis on the ability of fictitious play to provide a foundation for Nash equilibrium.

Fictitious play was originally proposed by Brown (1951) as an algorithm for computing equilibria of matrix games. The method can be interpreted as a multistage learning process, in which two players play the same game repeatedly and choose strategies in each period that are myopic best replies to their beliefs, which are given by the empirical distributions of their opponents' past strategy choices. It can also be considered as a form of Bayesian learning with special priors.[2] As a learning process fictitious play has many appealing features. It is simple but utilizes more information than the best-reply dynamics, in which at any instant players choose strategies that are best replies to the actions used by their opponents in the most recent play of the game. It can be represented by simple iterative formulae which can be easily analysed by a computer to trace the dynamics of the learning process. Most important of all, if fictitious play converges, it must be to a Nash equilibrium.[3]

I characterize sufficient conditions for 3×3 symmetric games under which beliefs generated by fictitious play will always converge to some Nash equilibrium. These conditions serve to delineate a class of 3×3 games for which equilibrium analysis can be justified by a learning theory. I also show that if player start with identical initial beliefs then the pair of beliefs either converges to a Nash equilibrium or approaches some limit cycles. Finally, replacing fictitious play by another more sophiscated learning process is shown to yield stronger convergence result.

Interest in the ability of fictitious play to provide a foundation for Nash equilibrium is not new. Robinson (1951) showed that fictitious play converges for all two-players zero-sum games, and Miyasawa (1961) obtained convergence result for all 2×2 games. These early results inspired a hope that fictitious play would converge for all two-players games. However, Shapley (1964) came up with an example of 3×3 game in which beliefs generated by fictitious play do not converge but instead approach limit cycles. Shapley's example indicates that fictitious play cannot provide a learning foundation for Nash equilibria in all 3×3 games, let alone general two-players games.

Recent studies of fictitious play have turned to the question of identifying the most general class of games in which fictitious play converges. Milgrom and Roberts (1990) and Krishna (1992) provide a partial answer to this question by showing that fictitious play converges in all games that satisfy strategic complementarities and diminishing marginal returns to increasing strategies. In light of the result of Krishna (1992), any 3×3 game that satisfies strategic complementarity and diminishing marginal returns in increasing strategies must yield convergence for fictitious play. The class of 3×3 symmetric games characterized in this chapter (in which fictitious play always converges) is a much wider class than (and includes) that implied by the two conditions. In particular, I do not require players' payoffs to subject to diminishing marginal returns and I allow for games that do not satisfy strategic complementarities.

For games that do not belong to this class, it would be useful to know the limiting dynamic behavior of players's beliefs. In fact, as pointed out by Mailath (1992), the appropriate description of behavior in situations where learning is unable to provide a satisfactory foundation for Nash equilibria will be dynamic. Assuming that players always start with the same belief, it can be shown that if fictitious play does not converge, then each player's belief must converge to some limit cycle. Similar result holds when the learning process is modified so that one player is allowed to have and act on the knowledge that his opponent is using historical distribution to predict his action in the present period. However, when additional knowledge is assumed of the players and they utilize such information rationally to maximize their one-period expected payoff learning inevitably leads to a pure-strategy Nash equilibrium within finite periods.

The rest of chapter 1 is organized as follow. Section 1.2 describes the structure of the game and the learning process. Section 1.3 presents the main result on convergence of fictitious play. Section 1.4 addresses the question of what happens when fictitious play does not converge to a Nash equilibrium. Section 1.5 investigates the consequences of introducing more knowledge and sophiscation into the learning process. Section 6 provides some suggestions for future research. Section 1.7 gives the conclusion.

1.2. The Game and the Learning Process

Consider any 3×3 symmetric game which is played repeatedly over time. Suppose the game starts at period 0 and at any period $t > 0$ each player chooses a pure strategy that maximizes his own expected one-period payoff with respect to his belief about his opponent's choice at period t. It is assumed that both players observe the choices of their opponents 'after' each play of the game. Players' beliefs are formed and updated according to a process called fictitious play, which is described as follow.

A belief is defined to be a probability distribution over the pure strategy space. Since there are only three pure strategies, a belief can be represented by a point in the unit simplex S_1^3. Denote the vertices of the simplex by δ_i, $i = 1, 2, 3$. They represent beliefs that assign probability 1 to one of the pure strategies and zero for others. At period 0 each player arbitrary chooses a pure strategy which becomes the belief of his opponent at period 1. For $t \geq 1$, let $p(t)$ $q(t) \in S_1^3$ be the beliefs of players 1 and 2 respectively at time t. Let $BR(p(t-1))$ and $BR(q(t-1))$ be their corresponding sets of pure-strategy best responses, given their beliefs. Furthermore, I adopt the tie-breaking rule that a player chooses the pure strategy with the largest label whenever he is indifferent between alternative strategies.

Denote the actual-response functions by $B(p(t))$ and $B(q(t))$ respectively, where $B(p(t)) = \max BR(p(t))$ and $B(q(t)) = \max BR(q(t))$. Given p(t) and q(t), beliefs at time t+1 are then defined by

$$p(t+1) = \frac{t}{t+1}\, p(t) + \frac{1}{t+1}\, \delta_{B(q(t))} \qquad (1)$$

$$q(t+1) = \frac{t}{t+1}\, q(t) + \frac{1}{t+1}\, \delta_{B(p(t))} \qquad (2)$$

(1) and (2) define the learning process for the game and has the convenient interpretation that the belief of a player at any time t is given by the historical distribution of his opponent's choices from 0 to t-1. It is well known that if p(t) converges to p and q(t) converges to q, then the pair (p, q) constitutes a mixed-strategy Nash equilibrium.[4] In the next section I characterize a subclass of 3×3 symmetric games for which beliefs generated by fictitious play inevitably converge.

1.3 Main Convergence Result

This section gives the result that for a reasonably wide class of

3×3 symmetric games fictitious play must converge to some Nash equilibrium. Before stating the main theorem it is necessary to introduce several lemmas that would be repeatedly used throughout this chapter. These lemmas capture many important properties of fictitious play in 3×3 games.

DEFINITION. Let p_1, p_2 belong to S_1^3. The interval $[p_1, p_2]$ is defined as the set of convex combinations of p_1 and p_2. That is,

$$[p_1, p_2] = \left\{ p \in S_1^3 \mid p = \alpha p_2 + (1-\alpha)p_1,\ 0 \leq \alpha \leq 1 \right\}$$

Furthermore, define

$$(p_1, p_2] = \left\{ p \in [p_1, p_2] \mid p \neq p_1 \right\},\ [p_1, p_2) = \left\{ p \in [p_1, p_2] \mid p \neq p_2 \right\}$$

and $(p_1, p_2) = (p_1, p_2] \cap [p_1, p_2)$.

DEFINITION. Let $E \subseteq S_1^3$. Define $B(E) = \left\{ B(p) \mid p \in E \right\}$. That is, $B(E)$ is the collection of all actual responses for the subset of beliefs E.

DEFINITION. A boundary $\partial(i, j)$, $i \neq j$, $i, j = 1, 2, 3.$, is a subset of beliefs in S_1^3 with the property that $p \in \partial(i, j)$ if and only $\{i, j\} \subseteq BR(p)$. More formally, define $BR^{-1}(i) = \left\{ p \in S_1^3 \mid i \in BR(p) \right\}$ $i=1,2,3$. Then $\partial(i, j) = BR^{-1}(i) \cap BR^{-1}(j)$. Note that $BR^{-1}(i)$ is convex and $\partial(i, j)$ is closed and bounded (compact).

DEFINITION. p_1 is said to stochastically dominates p_2 (in notation $p_1 \geq p_2$) if $\sum_{k=j}^{3} p_1[k] \geq \sum_{k=j}^{3} p_2[k]$ for $1 \leq j \leq 3$, where $p_1[k]$ and $p_2[k]$ are the probabilities assigned to pure strategy k by p_1 and p_2 respectively.

DEFINITION. A 3×3 symmetric game is said to have the monotone best-reply property if the pure-strategy set can be ordered in such a way that for any two mixed-strategies (or beliefs) p_1 and p_2, $p_1 \geq p_2$ implies $B(p_1) \geq B(p_2)$.

I like to emphasize that any game that satisfies strategic complementarity has the monotone bes' ːply property.[3] Therefore the set of games with the monotone best-reply property includes all games that satisfy strategic complementarities.

In the following lemmas, the results are unaffected when the roles of p(t) and q(t) are exchangeu. In addition, I use the unordered pair (p, q) to denote a pair of be'iefs in $S_1^3 \times S_1^3$ respectively. Thus when I say that beliefs p(t) and q(t) converge to an equilibrium $(\bar{p}, \bar{q})$ it should be understood that I mean either p(t) converges to $\bar{p}$ and q(t) converges to $\bar{q}$, or p(t) coiiverges to $\bar{q}$ and q(t) converges to $\bar{p}$.

Lemma 1. Define $B^{-1}(i) = \left\{p \in S_1^3 \mid B(p) = i\right\}$, i = 1, 2, 3. Consider any three pure strategies i, j and k, where $i \neq j$. If there exists a $\tilde{p} \in B^{-1}(i)$ such that $(\tilde{p}, \delta_k] \cap B^{-1}(j) \neq \emptyset$, $i \neq j$, then $[q, \delta_k] \cap B^{-1}(i) = \emptyset$ for all $q \in B^{-1}(j)$.

Proof. Suppose $\lambda q + (1-\lambda)\delta_k \in B^{-1}(i)$ for some $q \in B^{-1}(j)$ and some $\lambda \in [0, 1]$. Two possibilities arise. Either one has $j \notin BR(\lambda q + (1-\lambda)\delta_k)$ or $j \in BR(\lambda q + (1-\lambda)\delta_k)$.

Consider the first case. Since $j \in BR(q)$ j must be at least as good as i as a response against strategy q. If j is also at least as good as i against δ_k then j will be at least as good as i against any convex combination of q and δ_k. Since $i \in BR(\lambda q + (1-\lambda)\delta_k)$ this would imply that $j \in BR(\lambda q + (1-\lambda)\delta_k)$, which results in a contradiction. Therefore i must be strictly better than j against strategy δ_k. Now since $i \in BR(\bar{p})$ i must be at least as good as j against strategy $\bar{p}$. The last two statements imply that i must be strictly better than j against any $p \in (\bar{p}, \delta_k]$ and hence $[\bar{p}, \delta_k] \cap B^{-1}(j) = \emptyset$, which is a contradiction.

Now consider the second case where both i and j are best responses against the strategy $\lambda q + (1-\lambda)\delta_k$. Since i is the actual response against $\lambda q + (1-\lambda)\delta_k$ we must have $i > j$, which in turn implies that $i \notin BR(q)$ (since j is the actual response against q). With j a strictly better response against q than i it must be the case that i is at least as good against δ_k as j. Otherwise j would be a strictly better response than i against $\lambda q + (1-\lambda)\delta_k$. On the other hand since i is a best response against $\bar{p}$ this means that i must be at least as good as j against any convex combination of $\bar{p}$ and δ_k. But then one has $j \neq B(p)$ for any $p \in [\bar{p}, \delta_k]$ and hence $(\bar{p}, \delta_k] \cap B^{-1}(j) = \emptyset$, which is again a contradiction.

Q.E.D.

Lemma 2. Let i, j, k be distinct pure strategies. If $\partial(i, j) = \emptyset$, then for all $p \in BR^{-1}(i)$ and $q \notin BR^{-1}(j)$, we have $[p, q] \cap B^{-1}(j) = \emptyset$.

Proof. If $p, q \in BR^{-1}(i)$ then by convexity of $BR^{-1}(i)$ one has $[p, q] \subseteq BR^{-1}(i)$. Since $\partial(i, j) = \emptyset$ it is obvious that $[p, q] \cap B^{-1}(j) = \emptyset$.

So suppose for some $\dot{p} \in BR^{-1}(i)$ and $\dot{q} \in B^{-1}(k)\setminus(BR^{-1}(i) \cup BR^{-1}(j))$, $[\dot{p}, \dot{q}] \cap B^{-1}(j) \neq \emptyset$. Let

$$\dot{\alpha} = \sup \left\{ \alpha \in [0, 1] \mid \alpha\dot{p} + (1-\alpha)\dot{q} \in B^{-1}(j) \right\}$$

$$\underset{\cdot}{\alpha} = \inf \left\{ \alpha \in [0, 1] \mid \alpha\dot{p} + (1-\alpha)\dot{q} \in B^{-1}(j) \right\}$$

$$p^{*} = \dot{\alpha}\dot{p} + (1-\dot{\alpha})\dot{q} \in B^{-1}(j) \text{ and } q^{*} = \underset{\cdot}{\alpha}\dot{p} + (1-\underset{\cdot}{\alpha})\dot{q} \in B^{-1}(j).$$

Since $BR^{-1}(j)$ is closed we have $p^{*}, q^{*} \in BR^{-1}(j)$. Construct the sequence $p_n = \frac{1}{n}p + (1 - \frac{1}{n})p^{*}$, $n \geq 1$. Because $\partial(i, j) = \emptyset$, $p^{*} \neq p$ and by definition of p^{*} one has $p_n \notin BR^{-1}(j)$ for all n. Hence $p_n \in BR^{-1}(i) \cup BR^{-1}(k)$ for all n. Now for all n large enough we must have $p_n \notin BR^{-1}(i)$ otherwise $p^{*} \in BR^{-1}(i)$ by closedness of the latter. This implies that for all n large enough $p_n \in BR^{-1}(k) \backslash BR^{-1}(i)$.

Consider any $p_n \in BR^{-1}(k) \backslash BR^{-1}(i)$. Since $\dot{q} \in B^{-1}(k) \backslash (BR^{-1}(i) \cup BR^{-1}(j))$ k must be strictly better than j against any convex combination of p_n and $\dot{q}$. Consequently $[p_n, \dot{q}] \cap B^{-1}(j) = \emptyset$. Since $[p^{*}, q^{*}] \subseteq [p_n, \dot{q}]$ this results in a contradiction.

Q.E.D.

The geometrical meanings of lemmas 1 and 2 are best illustrated through examples. Consider the simplex in figure 1 which shows the distribution of best responses over different regions of the simplex. In this simplex $\bar{p}$ is a belief that belongs to $B^{-1}(2)$ and the line segment joining $\bar{p}$ and δ_1 intersects the region $B^{-1}(1)$. By letting i = 2, j = 1 and i = 1 it is obvious that the conditions of lemma 1 are satisfied and hence its conclusion must hold. In this case the lemma simply states the geometrical fact that the line segment joining any belief in $B^{-1}(1)$ and δ_1 will not cross the region $B^{-1}(2)$.

For lemma 2 consider the simplex in figure 2. Here we have the typical distribution of best responses for a 3×3 symmetric game with an asymmetric pure-strategy equilibrium. In figure 2 the boundary $\partial(2, 1)$

is empty. Now lemma 2 (in this case let $i = 2$ and $k = 1$) simply states the obvious fact that the line segment joining any belief in $B^{-1}(2)$ and δ_2 or δ_1 will never intersect the region $B^{-1}(1)$.

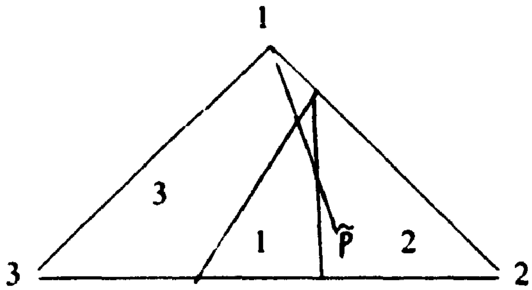

Figure 1

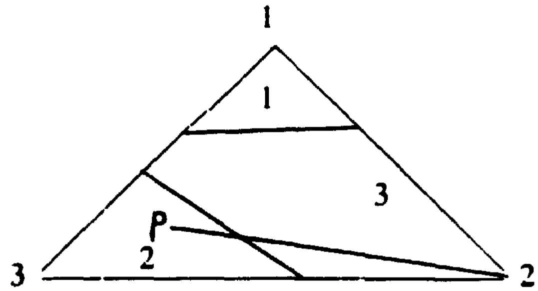

Figure 2

Lemma 3. Suppose there does not exist a $\bar{p} \in S_1^3$ with $BR(\bar{p}) = \{1, 2, 3\}$. Then there is a T such that for every $t > T$, whenever the following conditions are satisfied by the two sequences of beliefs:

(i) $[p(t), \delta_{B(q(t))}] \cap B^{-1}(j) \neq \emptyset$;

(ii) $j \neq B(p(t))$ and $j \neq B(\delta_{B(q(t))})$;

(iii) Given any $N > 0$, $B(p(t+k)) = B(p(t))$ for $1 \leq k \leq N$ implies $B(q(t+k)) = B(q(t))$ for $1 \leq k \leq N$. That is, starting from period t, as long as player 1 continues to use the strategy $B(p(t))$ player 2 will not change his strategy too,

there must exist a $t' > t$ such that $B(p(t+k)) = B(p(t))$ for $0 \leq k \leq t'-t-1$ and $B(p(t')) = j$.

Proof. See the Appendix.

I briefly explain the geometrical meaning of lemma 3. Consider players' beliefs at any time t. By definition of fictitious play player 1's belief (starting from period t) will be moving toward $\delta_{B(q(t))}$ and player 2's toward $\delta_{B(p(t))}$ until one of them changes his strategy. If the line segment joining p(t) and $\delta_{B(q(t))}$ (or q(t) and $\delta_{B(p(t))}$) crosses a region which entails a different actual response than B(p(t)) and B($\delta_{B(q(t))}$), and suppose player 1 (player 2) is the first to switch strategy, then two possibilities stand out. Either player 1's belief (player 2) falls into the region, in which case player 1 (player 2) will switch to a strategy which is the best response to any belief in that region, or it jumps across the region and he switches to B($\delta_{B(q(t))}$) (or B($\delta_{B(p(t))}$) for player 2).

Now lemma 3 states that if the game has no completely-mixed strategy equilibrium, then as long as t is large enough one can be sure that the former outcome will prevail. In other words, there will be no jump in belief across the region. This result enables me to eliminate a lot of confusing possibilities when analysing the dynamic behavior of beliefs.

When there is a completely-mixed strategy equilibrium, some modification to the lemma is required. Lemma 4 states that given any ε greater than 0, as long as t is sufficiently large and the line segment joining p(t) and $\delta_{B(q(t))}$ (or q(t) and $\delta_{B(p(t))}$) crosses a part of the region that does not lie within the ε-neighbourhood of $\bar{\bar{p}}$ player 1's (player 2's) belief will not jump acoss the region. Here ε is arbitrary but in general the smaller the ε, the larger t we need in order to have no jump.

Lemma 4. Suppose there exists a $\overline{p} \in S_1^3$ such that $BR(\overline{p}) = \{1, 2, 3\}$. Then for every $\varepsilon > 0$ there exists a $T(\varepsilon)$ such that whenever the two sequences of beliefs satisfy the following conditions:

(i) $[p(t), \delta_{B(q(t))}] \cap (B^{-1}(j)\backslash U_\varepsilon(\overline{p})) \neq \varnothing$, where $U_\varepsilon(\overline{p})$ is an ε-open ball containing $\overline{p}$ and

$$B^{-1}(j)\backslash U_\varepsilon(\overline{p}) = \left\{ p \in S_1^3,\ p \in B^{-1}(j) \text{ and } p \notin U_\varepsilon(\overline{p}) \right\};$$

(ii) $j \neq B(p(t))$ and $j \neq B(\delta_{B(q(t))})$;

(iii) Given any $N > 0$, $B(p(t+k)) = B(p(t))$ for $1 \leq k \leq N$ implies $B(q(t+k)) = B(q(t))$ for $1 \leq k \leq N$. That is, starting from period t, as long as player 1 continues to use the strategy $B(p(t))$ player 2 will not change his strategy too;

there corresponds a $t' > t$ so that $B(p(t+k)) = B(p(t))$ for $0 \leq k \leq t'-t-1$ and $B(p(t')) = j$.

Proof. See the appendix.

Lemma 5. Suppose there exists a T such that for all $t > T$, one has

(i) $B(p(t)) \in \{i_1, i_2\}$, $B(q(t)) \in \{j_1, j_2\}$, $i_1 \neq i_2$, $j_1 \neq j_2$;

(ii) $B(p(t)) = i_1$ if and only if $B(q(t)) = j_1$,

then $\{(p(t), q(t))\}$ must converge to some Nash equilibrium.

Proof. See the appendix.

Lemma 5 shows that if, starting from period t, none of the players uses uses more than two different pure strategies , then beliefs must converge to some Nash equilibrium. This result is extremely useful and plays a central role in the proof of the main theorem.

Let $A(p) = \left\{ j \in \{1,2,3\} \quad \delta_j \in BR^1(B(p)) \right\}$. That is, $A(p)$ consists of all pure strategies with the property that the actual response to p is also a best response to these strategies. The next lemma provides a sufficient condition under which beliefs will converge to a pure-strategy equilibrium.

Lemma 6. Suppose $B(p(t^*)) \in A(q(t^*))$, $(q(t^*)) \in A(p(t^*))$ at some time t^* and that either $B(p(t^*)) \neq 2$ or $B(q(t^*)) \neq 2$, or both. Then $\{(p(t), q(t))\}$ will converge to a pure-strategy Nash equilibrium.

Proof. See the appendix.

Lemma 6 says that if the players are playing a pure-strategy Nash equilibrium at some time t^* then as long as one of the players is not using pure strategy 2 beliefs must eventually converge to some pure-strategy equilibrium. I want to emphasize that the equilibrium to which beliefs converge may be different from the one played at t^*.The qualification that at least one player is not using strategy 2 is needed because there are games which have (2, 2) as a pure-strategy equilibrium but for which $B(p(t^*)) = B(q(t^*)) = 2$ at some time t^* does not necessary imply that beliefs will converge to a pure-strategy equilibrium. An example is given in figure 3.

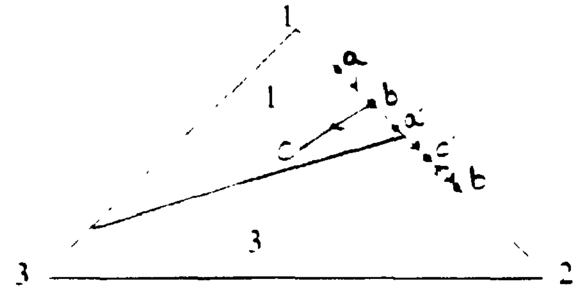

Figure 3

In figure 3 the best-response region $BR^{-1}(2)$ conicides with the boundary $[\delta_1, \delta_2]$ of the simplex. Suppose at some time t^* $p(t^*) = a$ and $q(t^*) = a'$ so that $B(p(t^*)) = B(p(t^*)) = 2$. Let t_1^* and t_2^* be the next two times when the pair of actual responses changes, so that $p(t_1^*) = b$, $q(t_1^*) = b'$; and $p(t_2^*) = c$, $q(t_2^*) = c'$. At t_2^* one has $B(p(t_2^*)) = 1$ and $B(q(t_2^*)) = 3$. Now it is possible that for all remaining periods the two players always change their strategies simultaneously and they alternate between strategies 1 and 3 infinitely often. When this happens beliefs will converge to a mixed-strategy equilibrium (which is an immediate consequence of lemma 5).

For any $p^* \in S_1^3$ and δ_i such that $B(p^*) \neq B(\delta_i)$ and $B([p^*, \delta_i]) = \{B(p^*), B(\delta_i)\}$, define $\phi(p^*, \delta_i) = \inf_q |p^* - q|$ (where the infinmum is taken over the set $[p^*, \delta_i] \cap B^{-1}(B(\delta_i))$) and $\tau(p^*, \delta_i) = \phi(p^*, \delta_i)/|p^* - \delta_i|$. That is, $\phi(p^*, \delta_i)$ is the length of that segment of $[p^*, \delta_i]$ that lies in the region $B^{-1}(p^*)$ while $\tau(p^*, \delta_i)$ is the ratio of this length to the total length of the interval.

Lemma 7. Consider any $p, q \in S_1^3$ where

(1) $B(p) \neq B(\delta_{B(q)})$, $B(q) \neq B(\delta_{B(p)})$; and

(2) $B([p, \delta_{B(q)}]) = \{B(p), B(\delta_{B(q)})\}$ and $B([q, \delta_{B(p)}]) = \{B(q,$

$B(\delta_{B(p)})$}.

(3) $\tau(p, \delta_{B(q)}) + \mu < \tau(q, \delta_{B(p)})$.

There exists a $T(\mu)$ such that for any $t' > T(\mu)$, if $p(t') = p$ and $q(t') = q$ then $B(p(t'')) = B(\delta_{B(q)})$ and $B(q(t'')) = B(q)$.

(i.e. t'' is the first time (after t') when one of the players changes his strategy (i.e. $B(p(t)) = B(p(t')) = B(p)$ and $B(q(t)) = B(q(t')) = B(q)$ for $t' \le t < t''$, whereas $B(p(t'')) \neq B(p(t'))$ or $B(q(t'')) \neq B(p(t'))$.)

Proof. See the appendix.

Lemma 7 gives the important result that by comparing two ratios one may be able to predict which player will switch his strategy first, provided t is large enough. If such a switch results in a pair of actual responses that satisfies the condition in lemma 6, then it follows immediately that $p(t)$, $q(t)$ will converge to a Nash equilibrium. Of course to apply the lemma one needs to know the relative magnitudes of the two ratios in question. Such information is usually made available through the fact that p and q fall within small neighbourhoods of some other beliefs p_0 and q_0 respectively where $\phi(p_0, \delta_{B(q)}) < \phi(q_0, \delta_{B(p)})$.

Lemma 8. Let $p(t') \in U_\varepsilon(p_0)$ for some $t' \ge 1$. Given any $N > 0$, suppose $p(t'+S) \in U_\varepsilon(p_0)$ for $0 \le S \le N-1$. Then we have

$$\sum_{S=1}^{N} \|p(t'+S) - p(t'+S-1)\| \le (\ln(t'+N+1) - \ln(t'+2) + \frac{1}{t'+1})(\varepsilon + M)$$

where $M = \max_j \|p_0 - \delta_j\|$.

Proof Consider any $p \in U_\varepsilon(p_0)$. We have for $i = 1, 2, 3$,

$$\|p - \delta_i\| \leq \|p - p_0\| + \|p_0 - \delta_j\|,$$

$$< \varepsilon + \max_j \|p_0 - \delta_j\| = \varepsilon + M.$$

Now since $p(t'+S) \in U_\varepsilon(p_0)$ for $0 \leq S \leq N-1$, we have

$$\sum_{S=1}^{N} \|p(t'+S) - p(t'+S-1)\| = \sum_{S=1}^{N} \frac{1}{t'+S} \|p(t'+S-1) - \delta_{B(q(t'+S-1))}\|$$

$$< \sum_{S=1}^{N} \frac{1}{t'+S} (\varepsilon + M)$$

$$< \left(\int_2^{N+1} \frac{1}{t'+x}\, dx + \frac{1}{t'+1}\right)(\varepsilon + M)$$

$$= (\ln(t'+N+1) - \ln(t'+2))(\varepsilon + M) + \frac{1}{t'+1} (\varepsilon + M),$$

which proves the lemma.

Q.E.D.

Lemma 8 states that the total distance moved by a player's belief within a neighbourhood $U_\varepsilon(p_0)$ between periods t' and $t' + N$ is bounded by some number that depends only on t', N, ε and the maximum distance between p_0 and one of the vertices of S_i^3.

Lemma 9. Let $\{p(t_k)\} \subseteq \{p(t)\}$ and $\{q(t_k)\} \subseteq \{q(t)\}$ be two subsequences of beliefs such that $p(t_k) \to p_0$ and $q(t_k) \to q_0$. Suppose (p_0, q_0) is not a Nash equilibrium. Then for all $\varepsilon, \delta > 0$ we can find $\delta_1 < \delta$, $\varepsilon_1 <$ ε and a $T(\varepsilon,\varepsilon_1,\delta,\delta_1)$ such that for all $t > T(\varepsilon)$, $q(t+S) \in U_{\delta}(q_0)$ for $0 \leq S \leq N$ implies $p(t+S) \in U_\varepsilon(p_0)$ for $0 \leq S \leq N$.

Proof. See the appendix.

An intuitive interpretation of lemma 9 is as follows. Consider any $(p_0, q_0) \in S_1^2 \times S_1^2$ which is not a Nash equilibrium and where the best response set of p_0 (or q_0) contains at most two pure strategies. Suppose (p_0, q_0) is a limit point of the sequence $\{p(t), q(t)\}$. Then given any $\varepsilon, \delta > 0$ one can always find neighbourhoods $U_{\varepsilon_1}(p_0)$ and $U_{\delta_1}(q_0)$ with radii strictly less than ε and δ respectively and which have the following property: for t large enough if player 1's belief happens to lie in $U_{\varepsilon_1}(p_0)$ and player 2's in $U_{\delta_1}(q_0)$, then the latter will leave the neighbourhood $U_{\delta_1}(q_0)$ before the former leaves $U_{\varepsilon}(p_0)$.

Theorem 1. Consider any 3×3 symmetric game. If the game satisfies one of the following two conditions:

(A) The game has an asymmetric pure-strategy Nash equilibrium; or

(B) The game has the monotone best-reply property;

then any sequence of beliefs $\{(p(t), q(t))\}_{t=1}^{\infty}$ that satisfy (1) and (2) must converge to some Nash equilibrium.

The formal proof, which is very lenghty, is given in the appendix. Here I will explain the general ideas and intuition behind the proof.

The proof goes by showing that for any game that satisfies either condition (A) or (B), fictitious play will lead to one of the following three outcomes:

(1) There exists a $T \geq 1$ such that for all $t > T$, $B(p(t)) = i$ and

$B(q(t)) = j$. That is, after time T both players will not change their strategies (or actual responses) again and beliefs converge to a pure-strategy equilibrium;

(2) There exists a T such that for all $t > T$, both players always change their strategies at the same time, and each of them alternates only between two pure strategies. In this case beliefs will converge to a mixed-strategy equilibrium;

(3) Every pure strategy is used by each player infinitely many times. In this case beliefs will converge to a competely-mixed strategy equilibrium.

Consider any game that satisfies condition (A). Without loss of generality let the asymmetric pure-strategy equilibrium be (δ_2, δ_3). Note that there are totally six feasible pairs of actual responses: (1, 1), (2, 2), (3, 3), (1, 2), (1, 3) and (2, 3). First single out those pairs the occurrence of which at any time $t \geq 1$ invariably implies convergence of beliefs to a pure-strategy equilibrium.

Denote by $(i(t), j(t))$ the unordered pair of strategy choices at time t (i.e. $i(t) = B(p(t))$ and $j(t) = B(p(t))$; or $i(t) = B(q(t))$ and $j(t) = B(p(t))$). It follows immediately from lemma 5 that if $(i(t), j(t)) = (2, 3)$ for some $t \geq 1$ then beliefs will converge to the equilibrium (δ_2, δ_3). If the possibility is eliminated of occurrence of the pair (2, 3) then for all $t \geq 1$, one has either $(i(t), j(t)) = (1, 1)$, (1, 2), (2, 2), (3, 3) or (1, 3). If in addition (δ_1, δ_1) is a pure-strategy equilibrium then again it follows that $(i(t), j(t)) = (1, 1)$ for some $t \geq 1$ will lead to convergence of beliefs to a pure-strategy equilibrium and therefore one can also eliminate the pair

(1, 1) from the list.

Depending on the geometrical distribution of best responses over the belief space S_1^2 some other pairs may also found to lead to convergence of beliefs to a pure-strategy equilibrium. For example, consider the game in figure 4.

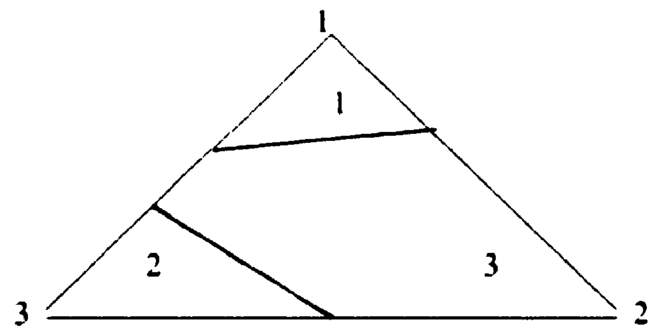

Figure 4

It is easy to see that if $(i(t), j(t)) = (1, 2)$ for some $t \geq 1$ then one would have $(i(t'), j(t')) = (2, 3)$ for some $t' > t$. At time t a player's belief is in the region $B^{-1}(2)$ and the other's in the region $B^{-1}(1)$. Starting at t, the former will move toward the vertex δ_1 along the line segment connecting the two while the latter will move along the line segment connecting it and δ_2. Because the first segment lies entirely within the region $B^{-1}(2)$ the first player will not change his strategy as long as the second player continues to use 1 as his actual response. On the other hand, the second segment intersects the region $B^{-1}(3)$ but not $B^{-1}(2)$ (this is obvious from the diagram, but to show that this is true for a subclass of games in a formal manner one has to invoke lemma 1) and so sooner or later the second player is going to switch his actual response to 3. When this happens one will get the pair of actual responses (2, 3).

After deleting every pair of actual responses that would certainly lead the beliefs to converge to a pure-strategy equilibrium one would be left with a subset of actual responses. If only one pair left one can immediately apply lemma 5 to show that beliefs converge to a mixed-strategy equilibrium. Otherwise it can be shown that either:

(i) there exists some T large enough so that for all $t \geq T$, at most two pairs of actual responses are feasible and each player will not use more than two different actual responses (in which case one has outcome 2): or

(ii) three or more pairs of actual responses occur infinitely many times (in which case one has outcome 3).

For games in which the intersection $\partial(1, 2) \cap \partial(1, 3) \cap \partial(2, 3)$ is empty (i.e. there does not exist a $\bar{\bar{p}}$ such that $BR(\bar{\bar{p}}) = \{1, 2, 3\}$) it can shown that only (i) is possible.

As an illustration consider the game in figure 5.

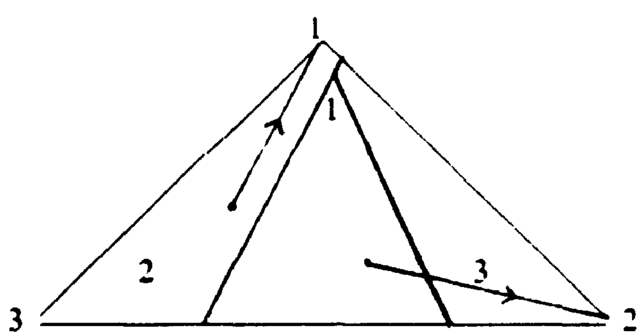

Figure 5

For this game the feasible pairs of actual responses (after eliminating those that would certainly lead to a pure-strategy equilibrium) are (2, 2), (3, 3), (1, 2) and (1, 3). If $(i(t'), j(t'))$ = (2, 2) for some t' then both players' beliefs will fall within the region $B^{-1}(2)$ at time t'. Starting at t' the two beliefs will move along two line segments connecting them and δ_2 respectively. Since the boundary $\partial(1, 2)$ is empty, by lemma 2 both segments will not intersect the region $B^{-1}(1)$. Let $t'' > t'$ be the time when one or both of the players changes his actual response. Because the pair (2, 3) has been eliminated from the list, one must have $(i(t''), j(t'')) = (3, 3)$. Starting at t'', beliefs will move toward the vertex δ_3. Since $[\delta_1, \delta_3] \cap B^{-1}(3) \neq \emptyset$ and $B(\delta_1) = 1 \neq 3$, by lemma 1 the two line segments joining the beliefs and δ_3 do not intersect $B^{-1}(1)$. Let $t_1 > t''$ be the next time one or both of the players changes his actual response. Again the two players must switch their strategies simultaneously and one has $(i(t_1), j(t_1)) = (2, 2)$, which is the same situation as one has at t'. By repeating the same argument above one can easily see that the two players always change their strategies together and they use only strategies 2 and 3.

Now suppose $(i(\bar{t}), j(\bar{t})) = (1, 3)$ for some $\bar{t}$. Starting at time t one player's belief (which lies in the region $B^{-1}(3)$) will move along the line segment connecting it and δ_1 and the other's along the segment connecting it and δ_3. The first segment intersects regions $B^{-1}(3)$ and $B^{-1}(1)$ only while the second one intersects $B^{-1}(1)$, $B^{-1}(3)$ and $B^{-1}(2)$, with $B^{-1}(3)$ in between. Since (1, 1) is a pair that has been ruled out, the first player cannot be the first to change his actual response. So either the second player changes his strategy first or both players change their strategies at the same time. In the first case the pair of actual responses will change from (1, 3) to (3, 3). In the second case the pair of actual responses will remain the same, provided $\bar{t}$ is large enough so that the second player's belief would not jump to $B^{-1}(2)$. Continuing in this manner, one has either $(i(t'), j(t')) = (3, 3)$ for some $t' > \bar{t}$ or $(i(t), j(t)) = (1, 3)$ for all $t > \bar{t}$.

Using similar argument, one can show that if $(i(\bar{t}), j(\bar{t})) = (1, 2)$ for some $\bar{t} > \hat{T}$ then $(i(t'), j(t')) = (1, 3)$ or $(3, 3)$ for some $t' > \bar{t}$. Behavior of beliefs subsequent to t' follows one of those described above.

For games in which $\partial(1, 2) \cap \partial(1, 3) \cap \partial(2, 3) = \{\bar{p}\}$–where $\bar{p}$ is an element in the belief space S_1^2 whose best-response set consists of all three pure strategies–one may have (i) or (ii), depending on whether or not $\bar{p}$ is a limit point of either $\{p(t)\}$ or $\{q(t)\}$.

If $\bar{p}$ is not a limit point of $\{p(t)\}$ and $\{q(t)\}$ then there exists an ε-neighbourhood containing $\bar{p}$ and a $\hat{T}$ so that for all $t > \hat{T}$, p(t) and q(t) always lie outside the neighbourhood. One can then apply lemma 4 and find another $T(\varepsilon) > \hat{T}$ such that for all $t > T(\varepsilon)$, there will be no jump in beliefs across any region that does not fall within the neighbourhood. Now the strategy of the proof follows closely that used for games without a completely-mixed strategy equilibrium. The only difference is that one have to treat the belief space as one with a hole $U_\varepsilon(\bar{p})$ so that beliefs never enter the hole.

If $\bar{p}$ is a limit point of $\{p(t)\}$ or $\{q(t)\}$ (i.e. there exists a subsequence $\{p(t_n)\}$ or $\{q(t_n)\}$ that converges to $\bar{p}$) it can be shown that (i) must prevail. First note that any pair $(\bar{p}, \bar{q})$ where $\bar{q} \neq \bar{p}$ cannot be a limit point. The proof involves an application of lemma 7 and lemma 9 and its basic idea can be illustrated through an example. Consider the game in figure 6

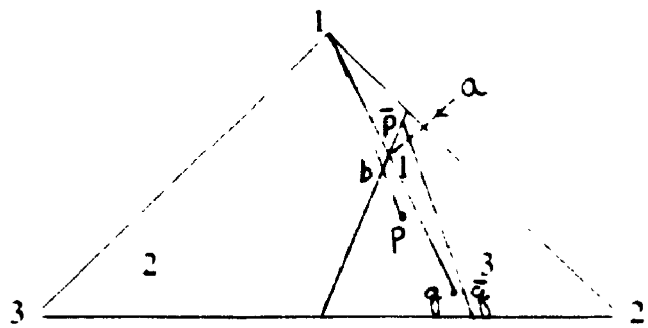

Figure 6

In figure 6 $\bar{q}$ is a point on the boundary $\partial(1, 3)$. For any p close to $\bar{p}$ and any q close to $\bar{q}$ such that $B(p) = B(q) = 1$, the ratio of the length of that part of $[p, \delta_1]$ that lies in the region $B^{-1}(1)$ to the total length of $[p, \delta_1]$ (i.e. $\overline{p\ a}/\overline{p\ \delta_1}$ in figure 6) is always smaller than the ratio of the length of that part of $[q, \delta_1]$ that lies in the region $B^{-1}(1)$ to the total length of $[q, \delta_1]$ (i.e. $\overline{q\ b}/\overline{q\ \delta_1}$ in figure 6). By lemma 7 if $p(t) = p$ and $q(t) = q$ for some t large enough then the player 1's belief will leave the region $B^{-1}(1)$ first and one has $(i(t'), j(t')) = (1, 2)$ for some $t' > t$.

Suppose $(\bar{p}, \bar{q})$ is a limit point of $\{(p(t), q(t))\}$. Using lemma 9 one can show that for some t' large enough (for lemma 7 to be applicable) p(t') and q(t') lie in the region $B^{-1}(1)$ and are very close to $\bar{p}$ and $\bar{q}$ respectively. Consequently $(i(t''), j(t'')) = (1, 2)$ for some $t'' > t'$. But (1, 2) is a pair of responses the occurrence of which implies that beliefs will eventually converge to the pure-strategy equilibrium (δ_2, δ_1). This contradicts the hypothesis that $\bar{p}$ is a limit point of $\{p(t)\}$ or $\{q(t)\}$.

Thus the only pair of limit points involving $\bar{p}$ is $(\bar{p}, \bar{p})$. To show that beliefs actually converge to this equilibrium consider any any ε-

neighbourhood of $\bar{p}$. By some tedious argument, it can be shown that there exist an ε'-neighbourhood of $\bar{p}$ contained in the former neighbourhood, and a $T(\varepsilon, \varepsilon')$ such that for all $t > T(\varepsilon, \varepsilon')$, if $p(t)$ and $q(t)$ fall into the the second neighbourhood then subsequent beliefs will never leave the first neighbourhood. Since ε is arbitrary this implies that beliefs converge to the equilibrium.

This completes the investigation of the class of games that satisfies condition (A), the strategy of the proof for games satisfying condition (B) is exactly the same (except that we eliminate different pairs of actual responses at the beginning) and will not be discussed.

I present some examples to illuatrate Theorem 1.

Example 1. Consider a 3×3 symmetric game with the following payoff matrix:

I \ II	1	2	3
1	(a_1, a_1)	(c_2, c_1)	(c_3, b_1)
2	(c_1, c_2)	(b_2, b_2)	(a_3, a_2)
3	(b_1, c_3)	(a_2, a_3)	(b_3, b_3)

where $a_i > b_i > c_i$, $i = 1, 2, 3$. This game has an asymmetric pure-strategy equilibrium (δ_2, δ_3) and a symmetric pure-strategy equilibrium (δ_1, δ_1). By Theorem 1 fictitious play will always converge to some Nash equilibrium. Suppose $p(1) = \delta_1$ and $q(1) = \delta_2$. As $B(\delta_1) = 1$ and $B(\delta_2) = 3$ $p(t)$ will move toward δ_3 and $q(t)$ will move toward δ_1. The movements continue until one of the players changes his strategy.

If $(a_2-c_2)/(a_1-b_1+a_2-c_2) = (a_1-c_1)/(a_1-c_1+a_3-c_3)$ then both players will switch their strategies simultaneously. Immediately after this has happened $p(t)$ will start moving back to δ_3 and $q(t)$ moving back to δ_2. The movements continue until both players change their strategies again, at which time $p(t)$ and $q(t)$ will reverse their directions of movement. This process will repeat indefinitely as long as $p(t) = \bar{p}$ (and hence $q(t) = \bar{q}$) never occurs. Thus $p(t)$ always moves back and forth around $\bar{p}$ and $q(t)$ moves back and forth round $\bar{q}$. Moreover, $\|p(t) - \bar{p}\| < \frac{1}{t}$ and $\|q(t) - \bar{q}\| < \frac{1}{t}$ for all $t \geq 1$. Therefore $(p(t), q(t))$ must converge to the mixed-strategy equilibrium $(\bar{p}, \bar{q})$.

If $(a_2-c_2)/(a_1-b_1+a_2-c_2) > (a_1-c_1)/(a_1-c_1+a_3-c_3)$ (or $p(\bar{t}) = \bar{p}$ and $q(\bar{t}) = \bar{q}$ for some $\bar{t} > 1$) then for some $t' > 1$ one has $B(p(t')) = 2$ and $B(q(t')) = 3$ and beliefs will converge to the pure-strategy equilibrium (δ_2, δ_3).

If $(a_2-c_2)/(a_1-b_1+a_2-c_2) < (a_1-c_1)/(a_1-c_1+a_3-c_3)$ then for some $t' > 1$ one has $B(p(t')) = B(q(t')) = 1$ and beliefs will converge to the symmetric pure-strategy equilibrium (δ_1, δ_1).

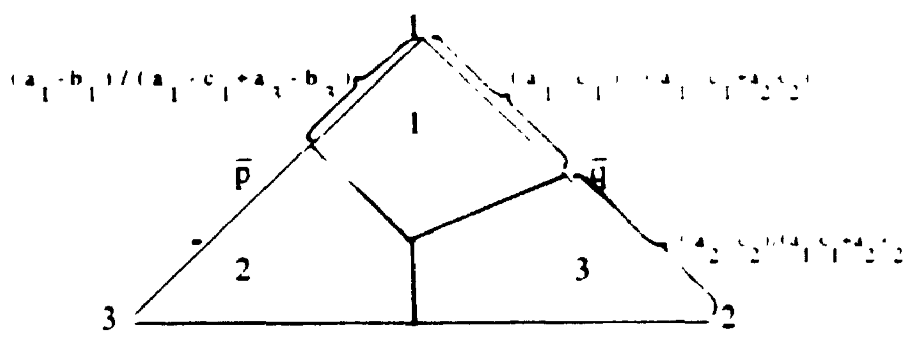

Figure 7

Now suppose $p(1) = q(1) = \delta_2$ or δ_3. In this case the two players will use identical strategies and they alternate between strategies 2 and 3 infinitely often. Consequently beliefs will converge to a mixed-strategy equilibrium.

Example 2. Consider again the 3×3 game in example 1, except that now it is assumed $c_1 > a_1 > b_1$, $a_2 > c_2 > b_2$, $a_3 > c_3 > b_3$ and $\frac{c_2-b_2}{a_2-c_2} < \frac{c_1-a_1}{a_1-b_1}$ and $\frac{a_2-c_2}{a_2-b_2} < \frac{c_3-b_3}{a_3-b_3}$. This game has the same asymmetric pure-strategy equilibrium as before but no symmetric pure-strategy equilibrium (see figure 8). Again by Theorem 1 any sequence of beliefs generated by fictitious play must converge to some Nash equilibrium.

Suppose $p(\bar{t}) = q(\bar{t})$ for some $\bar{t} \geq 1$ (say, $p(\bar{t}) = q(\bar{t}) = (1/3, 1/3, 1/3)$). We have $B(p(\bar{t})) = B(q(\bar{t}))$ and it follows from equations (1) and (2) (see section 2) that $p(\bar{t}+1) = q(\bar{t}+1)$, which in turn implies that $p(\bar{t}+2) = q(\bar{t}+2)$. Proceeding inductively one has $p(t) = q(t)$ for all $t \geq \bar{t}$. Therefore the sequence $\{(p(t), q(t))\}$ must converge to a symmetric Nash equilibrium. Since the only symmetric Nash equilibrium for this game is the completely-mixed strategy equilibrium $(\bar{p}, \bar{p})$, the pair of beliefs must converge to $(\bar{p}, \bar{p})$.

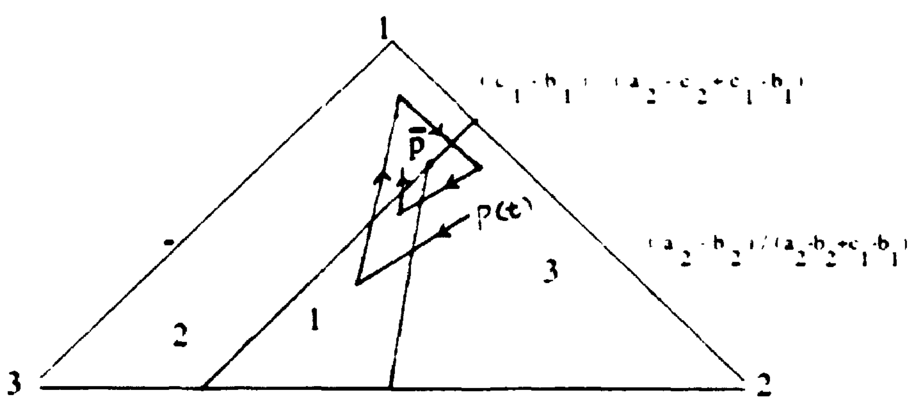

Figure 8

For the details, note that for any $t > \bar{t}$, $p(t+1)$ is a convex combination of p(t) and $\delta_{B(p(t))}$. This means that p(t) always moves toward the vertex $\delta_{B(p(t))}$ along the line segment connecting p(t) and $\delta_{B(p(t))}$. As the segment must cross a region with actual response different from B(p(t)) sooner or latter the belief will change its direction of movement. This being the case, p(t) will have to change its direction of movement infinitely many times. Figure 8 shows part of a hypothetical adjustment path of p(t). Tracing the dynamic adjustment path of p(t) graphically it can be easily seen that for any $i \in \{1, 2, 3\}$, $p(t) \in B^{-1}(i)$ for infinitely many ts. In addition, p(t) tends to move closer and closer to $\bar{p}$ as time goes by. Indeed, for any small open neighbourhood containing $\bar{p}$ there exists a T so large that starting at period T, once p(t) enters the neighbourhood it will never come out again. Thus p(t) converges to $\bar{p}$ and the pair of beliefs (p(t), q(t)) converges to the completely-mixed strategy equilibrium $(\bar{p}, \bar{p})$.

Example 3. Consider another 3×3 symmetric game with the following payoff matrix:

I \ II	1	2	3
1	(a, a)	(d, b)	(c, c)
2	(b, d)	(e, e)	(b, d)
3	(c, c)	(d, b)	(a, a)

where $a > b > c$, $d > e$, and $a + c < 2b$. Figure 9 shows the geometrical distribution of best responses for this game. It is easy to verify that this game has the monotone best-reply property. Therefore by Theorem 1 fictitious play will always converge. However, unlike example 2 beliefs will never converge to the unique completely-mixed strategy equilibrium. In fact, if $p(t') \in B^{-1}(1)$ and $q(t') \in B^{-1}(2)$ (or $p(t') \in B^{-1}(2)$ and $q(t') \in B^{-1}(1)$) for some $t' \geq 1$ then starting at t' $p(t)$ and $q(t)$ will move toward δ_2 and δ_1 respectively. As the line segment joining $p(t')$ and δ_2 lies entirely within the region $B^{-1}(1)$ while the segment joining $q(t')$ and δ_1 crosses the region $B^{-1}(1)$ player 2 will be the first one who changes his strategy. As soon as this happens both beliefs will be within the region $B^{-1}(1)$. Thereafter $p(t)$ and $q(t)$ will continue to move toward δ_1 and never change their directions of movement again. Similarly, if $p(t') \in B^{-1}(2)$ and $q(t') \in B^{-1}(3)$ (or $q(t') \in B^{-1}(2)$ and $p(t') \in B^{-1}(3)$) then sooner or latter both beliefs will show up in the region $B^{-1}(3)$ and eventually converge to δ_3. In both cases $(p(t), q(t))$ converges to a symmetric pure-strategy equilibrium.

On the other hand, if $p(1) = \delta_1$ and $q(1) = \delta_3$ ($p(1) = \delta_3$ and $q(1) = \delta_1$) then we have $p(2) = q(2) \in B^{-1}(2)$ and hence $p(t) = q(t)$ for all $t \geq 2$. Starting at period 2 both beliefs will move toward δ_2 until they emerge from the region $B^{-1}(2)$ simultaneously, at which time one has $p(t) = q(t) \in B^{-1}(3)$ and beliefs will converge to the symmetric pure-strategy equilibrium (δ_3, δ_3).

This result enables me to categorize all 3×3 symmetric games into two broad subclasses based on the outcomes of fictitious play.

For the rest of this section I assume that players always start with identical beliefs so that $p(1) = q(1)$. Using the two equations of evolution of beliefs (1) and (2) one has $p(2) = q(2)$, $p(3) = q(3)$,.... and so on. Consequently equations (1) and (2) can be reduced to:

$$p(t+1) = \frac{t}{t+1}\, p(t) + \frac{1}{t+1}\, \delta_{B(p(t))} \qquad t \geq 1 \qquad (3)$$

Equation (3) will replace (1) and (2) as the equation of evolution of beliefs.

The requirement that players start with the same initial beliefs may seem restrictive, but it can be justified on the ground that every 3×3 symmetric game can be reinterpreted as a game with a very large population of identical players (say, an infinite number of players) each of them has three pure strategies to choose from, and whose payoff is simply a linear combination of the fractions of population using different pure strategies. It follows that whatever the individual strategy choices are at period 0, after the initial play all players will observe the same aggregrate distribution of strategy choices and consequently they will have the same initial beliefs.

DEFINITION. A limit cycle C in the belief space S_1^2 is a finite union of intervals which satisfies the following conditions:

(i) It can be expressed in the form $C = [a_1, a_2] \cup [a_2, a_3] \cup .... \cup [a_m, a_{m+1}] \cup .... \cup [a_N, a_1]$;

(ii) For all $1 \leq m < N$, $B((a_m, a_{m+1})) = \{ i_m \}$ and $B((a_N, a_1)) = \{ i_N \}$.

Moreover, $[a_m, a_{m+1}] \subseteq [a_m, \delta_{i_m}]$ and $[a_N, a_1] \subseteq [a_N, \delta_{i_N}]$;

(iii) $a_i \neq \delta_k$, $1 \leq i \leq N$, $k = 1, 2, 3$.

Explained geometrically, a limit cycle is a closed geometrical figure consisting of a finite number of line segments such that one can trace the whole figure by moving continuously in the direction of actual responses, which does not change along any given segment (excluding the two end points). Condition (iii) of the definition restricts a limit cycle to be strictly inside the unit simplex. This condition has been added since beliefs generated by fictitious play will never converge to any cycle that involves the vertex/ices. Of course, whether a limit cycle exists or not is determined by the geometrical distribution of best responses in S_1^3 which in turn depends on the distribution of payoffs for the game.

It is easy to verify that any limit cycle must consist of at least three nonparallel line segments.

Lemma 10. Let C be a limit cycle of a 3×3 symmetric game. For any $i \in \{1, 2, 3\}$ there exists $[a_m, a_{m+1}] \subseteq C$, $1 \leq m \leq N$ (I adopt the convention that $a_{N+1} = a_1$) such that $B((a_m, a_{m+1})) = \{ i \}$.

Proof. Suppose there exists some $i^* \in \{1, 2, 3\}$ such that $(a_m, a_{m+1}) \cap B^{-1}(i^*) = \varnothing$ for $1 \leq m \leq N$. By virtue of the definition of a limit cycle one has $a_m[i^*] \geq a_{m+1}[i^*]$ and hence $a_1[i^*] = 0$ (i.e. the probability assigns to pure strategy i^* by belief a_1 must be 0). Let $M \leq N$ be such that $B((a_{M-1}, a_M)) \neq B((a_M, a_{M+1}))$. Since $a_M[i^*] = a_1[i^*] = 0$ we must have $(a_M, a_{M+1}) \cap (a_{M-1}, a_M) \neq \varnothing$. But this implies that $B((a_{M-1}, a_M)) =$

$B((a_M, a_{M+1}))$, which is a contradiction.

Q.E.D.

That a limit cycle exists does not necessarily imply that beliefs generated by fictitious play would converge in their dynamic behaviours to such a cycle. The following lemma gives sufficient conditions under which a limit cycle, if it exists, actually represents the asymptotic dynamic behaviour of beliefs.

DEFINITION. A sequence of beliefs $\{p(t)\}$ is said to converge to some limit cycle C if and only if $\lim_{t\to\infty} d(p(t), C) = 0$.

Lemma 11. Consider any 3×3 symmetric game in which the pure strategies can be labelled in such a way that $B(\delta_1)$, $B(\delta_2)$ and $B(\delta_3)$ are all distinct and $B(\delta_i) \neq i$ for $i = 1, 2, 3$. Suppose the game admits a limit cycle C. Then any sequence of belief $\{p(t)\}$ which evolves according to (3) must converge to the limit cycle C. Consequently, C is the game's unique limit cycle.

Proof. See the appendix.

Lemma 12 complements the result of lemma 11. Together the two lemmas guarantee that for a special class of 3×3 symmetric games, beliefs generated by fictitious play either converge to a limit cycle or to some Nash equilibrium.

Lemma 12. Consider any 3×3 symmetric game in which the pure strategies can be labelled in such a way that $B(\delta_1)$, $B(\delta_2)$ and $B(\delta_3)$ are all distinct and $B(\delta_i) \neq i$ for $i = 1, 2, 3$. Suppose the game does not admit any limit cycle. Then any sequence of beliefs which evolves according to (3) must converge to some Nash equilibrium.

Proof. See the appendix.

Theorem 2 gives the the main result of this section.

Theorem 2. Consider any 3×3 symmetric game. Then any sequence of beliefs $\{(p(t), q(t))\}_{t=1}^{\infty}$ which satisfies (3) will either converge to some symmetric Nash equilibrium or approach some pair of limit cycles.

Proof. Divide the class of 3×3 symmetric games into 3 exclusive subclasses:

(S1) the pure strategies can be labelled in such a way that $B(\delta_1)$, $B(\delta_2)$ and $B(\delta_3)$ are all distinct and $B(\delta_i) \neq i$, $i = 1, 2, 3$;

(S2) regardless of how the pure strategies are labelled, $B(\delta_1)$, $B(\delta_2)$ and $B(\delta_3)$ are not all distinct and $B(\delta_i) \neq i$ for $i = 1, 2, 3$;

(S3) regardless of how the pure strategies are labelled, $B(\delta_k) = k$ for some k.

From lemmas 11 and 12 it follows that for any game in subclass (S1), beliefs generated by (3) and (4) either converge to some Nash equilibrium or approach some limit cycle. Therefore one needs only consider subclasses (2) and (3).

Consider any game that belongs to (S2). Without loss of generality let $B(\delta_1) = B(\delta_2)$. Since $B(\delta_i) \neq i$ for $i = 1, 2, 3$ one must have $B(\delta_1) = B(\delta_2) = 3$ and $B(\delta_3) = 2$, which implies that (δ_2, δ_3) is an asymmetric pure-strategy equilibrium. It follows immediately from Theorem 1 that beliefs must converge to some Nash equilibrium.

Next consider any game in subclass (S3). If $B(p(t)) = B(q(t)) \neq k$ for all $t \geq 1$ then the two players use at most two pure strategies as actual responses and hence by lemma 5 beliefs must converge to some Nash equilibrium. So suppose $B(p(t')) = B(q(t')) = k$ for some $t' \geq 1$. Note that $B(p(t')) \in A(B(q(t')))$ and $B(q(t')) \in A(B(p(t')))$. If $k \neq 2$ then by lemma 6 $(p(t), q(t))$ will converge to some equilibrium. If $k = 2$ then since $B(p(t')) = B(q(t')) = B(\delta_2) = 2$ and $p(t'+1) = q(t'+1)$ is a convex combination of $p(t')$ and δ_2 we have $B(p(t'+1)) = B(q(t'+1)) = 2$, which in turn implies that $B(p(t'+2)) = B(q(t'+2)) = 2$. Continuing in this manner one gets $B(p(t)) = B(q(t)) = 2$ for all $t \geq t'$ and hence $(p(t), q(t))$ will converge to the pure- strategy equilibrium (δ_2, δ_2).

Q.E.D.

Two remarks are in order. First, as can be seen from the above proof, convergence of beliefs to a limit cycle is only possible within a very special subclass of games. Second, the limiting outcome of fictitious play does not depend on the initial beliefs as long as they are identical. In other words, the nature of the asymptotic behaviour of beliefs which evolve according to equation (3) is a well-defined property of a 3×3 symmetric game.

1.5 Alternative Learning Process

Fictitious play is a fairly naive learning process. The information requirement of fictitious play is minimal. Each player forms his belief solely on the basis of the statistical distribution of his opponent's previous strategy choices. Neither player is required to know his opponent's objective and payoff function or needs to have any idea about how his opponent uses his own past observations. The consequences of such a learning process in 3×3 symmetric games were explored in the previous sections. In this section I examine the effects of modifying the learning process to incorporate more information requirement and rationality on the part of the players.

I consider an alternative learning processe that differs from fictitious play in the amount of information available to and utilized by the players as they play the game repeatedly. The major concern here is how well this learning process is compared to fictitious play in providing a foundation for Nash equilibria in 3×3 symmetric games. It is natural to speculate that a learning theory that involves more information utilization should perform better than or at least as good as fictitious play in providing a foundation for Nash equilibria. The results we find actually confirm this speculation.

The informational endowments and learning behaviours underlying the learning process are specified below. Note that when a player is said to use his information rationally it means that he forms his belief by taking into account 'all' logical implications of his information. Also, in what follows the roles of player 1 and player 2 are chosen for convenient reference only and are interchangeable.

Learning Process I.

(i) player 1's belief evolves according to equation (1);

(ii) player 2 knows (i);

(iii) player 2 knows player 1's payoff function and he knows that player 1 chooses actual response to maximize his own one-period expected payoff;

(iv) player 2 uses his information rationally.

The major difference between fictitious play and learning process I is that in the latter one of the players knows how his opponent forms his belief and he acts on this knowledge. Since he also knows his opponent's payoff function and objective he is able, by using his information rationally, to accurately predict his opponent's strategy choice. Thus the equations of evolution of beliefs become:

$$p(t) = \frac{t-1}{t}\, p(t-1) + \frac{1}{t}\, \delta_{B(q(t-1))} \qquad t \geq 2 \qquad (1)$$

$$q(t) = \delta_{B(p(t))} \qquad t \geq 2 \qquad (2')$$

where equation (2') conveys the fact that player 2's belief at any time t coincides with his opponent's actual strategy choice at time t.

Proposition 1 shows that if a sequence of beliefs generated by learning process I converges, it must be to a pure-strategy equilibrium.

Proposition 1. Suppose the sequence $\{(p(t), q(t))\}$ defined by (1) and (2') converges to (p^*, q^*). Then (p^*, q^*) must be a pure-strategy equilibrium.

Proof. Suppose $p(t)$ converges to p^* and $q(t)$ converges to q^*. Since

$q(t) = \delta_{B(p(t))}$ by upper-hemi continuity of the best-response correspondence q^* must be a best response to p^*. On the other hand, since $q(t)$ is a point mass for all t q^* must be a distribution concentrated on a single pure strategy. This means that there exists a $T \geq 1$ such that for all $t > T$, $q(t) = q^*$. Therefore $B(q(t)) = B(q^*)$ for all $t > T$ and hence $p^* = \delta_{B(q^*)}$.

Q.E.D.

Define $\overline{BR}(p) = \left\{ B(\delta_i) \mid i \in BR(p) \right\}$ and $\overline{B}(p) = \max \{ j \mid j \in \overline{BR}(p) \}$ where $p \in S_1^3$. Also, define $\overline{BR}^{-1}(i) = \left\{ p \in S_1^3 \mid i \in \overline{BR}(p) \right\}$ and $\overline{B}^{-1}(i) = \left\{ p \in S_1^3 \mid \overline{B}(p) = i \right\}$, $i = 1, 2, 3$. It is not difficult to show that if $B(\delta_1)$, $B(\delta_2)$ and $B(\delta_3)$ are all distinct then both $\overline{BR}^{-1}(i)$ and $\overline{B}^{-1}(i)$ are convex sets.

Lemma 13. Suppose $B(\delta_1)$, $B(\delta_2)$ and $B(\delta_3)$ are all distinct. Then both $\overline{BR}^{-1}(i)$ and $\overline{B}^{-1}(i)$ are convex. Moreover, there exists a 3×3 symmetric game such that $\overline{BR}$ and $\overline{B}$ are the set of pure-strategy best responses and the actual response for the game respectively.

Proof. Let p_1, $p_2 \in \overline{BR}^{-1}(i)$. Since $B(\delta_1)$, $B(\delta_2)$ and $B(\delta_3)$ are all distinct there exists some $\bar{i} \in BR(p_1) \cap BR(p_2)$ such that $B(\bar{i}) = i$. By convexity of $BR^{-1}(\bar{i})$ one has $\bar{i} \in BR(\alpha p_1 + (1-\alpha)p_2)$ and hence $i \in \overline{BR}(\alpha p_1 + (1-\alpha)p_2)$ for all $\alpha \in [0,1]$. Similarly one can show that $\overline{B}^{-1}(i)$ is convex.

For the second part suppose the payoff function of a player in the original game is given by $U(i, j)$, where i and j represent the pure strategies used by the player and his opponent respectively. Define a new 3×3 symmetric game in which the payoff function of the same player is given by $\bar{U}(B(\delta_i), j) = U(i, j)$, $i, j = 1, 2, 3$. That is, if the player uses pure strategy $B(\delta_i)$ and his opponent uses pure strategy j he receives the same payoff as he does in the orginal game when he uses pure strategy i and his opponent uses j. Since $B(\delta_1)$, $B(\delta_2)$ and $B(\delta_3)$ are all distinct this new payoff function is well defined. It is easy to verify that $\overline{BR}$ and $\bar{B}$ give the set of best responses and the actual responses for this new game.

Q.E.D.

Notice that even when $B(\delta_1)$, $B(\delta_2)$ and $B(\delta_3)$ are all distinct $B(\delta_{B(p)})$ does not necessary equal to $\bar{B}(p)$. For example, if $B(\delta_1) = 2$ and $B(\delta_2) = 1$ and $BR(p') = \{1, 2\}$ then $B(\delta_{B(p')}) = 1$ but $\bar{B}(p') = 2$. However, one can always relabel the pure strategies so that $\bar{B}(p)$ defined on the relabelled set of strategies and $B(\delta_{B(p)})$ defined on the original labelled set will always correspond to the same pure strategy.

Combining the two equations (1) and (2') that jointly define learning process I, one gets

$$p(t) = \frac{t-1}{t}\, p(t-1) + \frac{1}{t}\, \delta_{B(\delta_{B(p(t-1))})}, \quad t \geq 2. \qquad (1')$$

Equation (1') gives the belief of player 1 at period t as a function of his belief at period t-1 alone. If $B(\delta_1)$, $B(\delta_2)$ and $B(\delta_3)$ are all distinct then by lemma 4 there exists a 3×3 symmetric game (with appropriate relabelling of pure strategies) such that $B(\delta_{B(p)})$ and $\bar{B}(p)$

(defined on the relabelled set of strategies) corresponds to the same pure strategy. Therefore (1') can be rewritten as

$$p(t) = \frac{t-1}{t}\, p(t-1) + \frac{1}{t}\, \delta_{\overline{B(p)}} \qquad t \geq 2. \tag{1''}$$

Equation (1") resembles equation (3) in section 4 and one can immediately apply the results of that section.

Proposition 2. Suppose $B(\delta_1)$, $B(\delta_2)$ and $B(\delta_3)$ are all distinct. Then $p(t)$ either converges to some $\bar{p} \in S_1^3$ or approaches some limit cycle.

Proof. This follows immediately from lemma 13 and Theorem 2.

Q.E.D.

Unfortunately, convergence of $p(t)$ to a single point does not automatically imply that $(p(t), q(t))$ will converge to a Nash equilibrium. As an example, consider a 3×3 symmetric game in which $BR(\delta_1) = \{ 2 \}$, $BR(\delta_2) = \{ 3 \}$ and $BR(\delta_3) = \{ 1 \}$. This game has no pure-strategy equilibrium. Now suppose $p(t)$ converges to some $\bar{p}$ in S_1^3. By Proposition 1 if $q(t)$ also converges then the pair $(p(t), q(t))$ must converge to a pure-strategy equilibrium, which is clearly impossible.

One might conjecture that even though $q(t)$ does not converge the empirical distribution of player 1's strategy choices might itself converge to some distribution $\bar{q}$ so that the pair $(\bar{p}, \bar{q})$ constitutes a Nash equilibrium. Again this need not be true. In the above example if the components of $\bar{p}$ are all positive but different, then the statistical distribution of player 1's strategy choices can be shown to converge to some $\bar{q}$ with positive and different components. In fact, let $\bar{p} = (\bar{p}[1], \bar{p}[2], \bar{p}[3])$ and we have $\bar{q} = (\bar{p}[3], \bar{p}[1], \bar{p}[2])$. Since $(\bar{p}, \bar{p})$ is the unique completely-mixed strategy equilibrium the pair $(\bar{p}, \bar{q})$ cannot be an equilibrium.

Although the sequence of beliefs defined by (1) and (2') does not always converge, one can still, as in the case of fictitious play, identify a reasonably wide class of games in which it does. Proposition 3 gives the important result that beliefs generated by learning process I converge in the same class of games investigated in section 1.3.

Proposition 3. Consider any 3×3 symmetric game. If the game satisfies one of the following two conditions:
(A) The game has an asymmetric pure-strategy Nash equilibrium; or
(B) The game has the monotone best-reply property;
then any sequence of beliefs $\{(p(t), q(t))\}_{t=1}^{\infty}$ that satisfy (1) and (2') must converge to some pure-strategy Nash equilibrium.

Proof. By Proposition 1 it suffices to show that beliefs converge in any game that satisfies condition (A) or (B). Consider any game that satisfies condition (A). Without loss of generality let the asymmetric Nash equilibrium be (δ_2, δ_3). I further distinguish between the following four possibilities.

(A1) $B(\delta_3) = 2$ and $B(\delta_1) = 1$;

(A2) $B(\delta_3) = 2$ and $B(\delta_1) = 2$;

(A3) $B(\delta_3) = 2$ and $B(\delta_1) = 3$;

(A4) $B(\delta_3) = 3$.

Suppose one has (A1). If there is a t' such that B(p(t')) = 1 then by hypothesis $B(\delta_{B(p(t'))}) = 1$. Since from (1') p(t'+1) is a convex combination of p(t') and $\delta_{B(\delta_{B(p(t'))})}$, by convexity of $B^{-1}(1)$ one has B(p(t'+1)) = 1. Continuing in this manner one gets B(p(t)) = 1 and hence $q(t) = \delta_{B(p(t))} = \delta_1$ for all $t \geq t'$. Thus both p(t) and q(t) converge to δ_1.

If there is a t' such that B(p(t')) = 2 then using the same reasoning as above one can show that B(p(t'+1)) = 2 for all $t \geq t'$. Therefore $q(t) = \delta_{B(p(t))} = \delta_2$ for all $t \geq t'$ and p(t) converges to δ_1. Similarly it can be shown that if there is t' such that B(p(t')) = 3 then $q(t) = \delta_3$ for all $t \geq t'$ and p(t) converges to δ_2.

Next consider possibility (A2). Suppose there is a t' such that $B(\delta_{B(p(t'))}) = 3$. Then it follows that B(p(t')) = 2 and by convexity of $B^{-1}(2)$ one has B(p(t'+1)) = 2. By repeating the same argument one gets B(p(t)) = 2 for all $t \geq t'$ and therefore $q(t) = \delta_2$ for all $t \geq t'$, which in turn implies that p(t) converges to δ_3.

On the other hand, suppose $B(\delta_{B(p(t))}) = 2$ for all $t \geq 1$. Then starting from period 1 p(t) will be moving towards δ_2 and it never changes direction. By convexity of B^{-1} there exists a $p' \in [p(1), \delta_2)$ such that $B([p', \delta_2))$ is a singleton. Therefore B(p(t)) will stay the same for all t large enough and hence $q(t) = \delta_{B(p(t))}$ must converge.

The proof of (A3) is very similar to that of (A2). In fact, if there exists a t' such that $B(\delta_{B(p(t))}) = 2$ then p(t) will converge to δ_2 and q(t) will converge to δ_3. On the other hand, if $B(\delta_{B(p(t'))}) = 3$ for all t then p(t) will converge to δ_3 and q(t) will converge to δ_1 or δ_2.

Finally consider possibility (A4). If $B(\delta_1) = 3$ then $(p(t), q(t))$ will certainly converge to (δ_3, δ_3). If $B(\delta_1) \neq 3$ and $B(\delta_{B(p(t'))}) = 3$ for some $t' \geq 1$ we must have $B(p(t')) = 2$ or 3. Since both 2 and 3 are best responses to δ_3 we have $B(p(t'+1)) = 2$ or 3. In either case $B(\delta_{B(p(t'+1))}) = 3$. Proceeding inductively one obtains $B(\delta_{B(p(t))}) = 3$ for all $t > t'$. Therefore $p(t)$ must converge to δ_3. Since starting at period t $p(t)$ moves towards δ_3 without changing its direction $B(p(t))$ must stay the same for all t large enough. Hence $q(t)$ must converge too.

If $B(\delta_1) \neq 3$ and $B(\delta_{B(p(t))}) = 1$ or 2 for all t then $p(t)$ will converge to δ_1 or δ_2. Again since $p(t)$ never changes its direction of movement $B(p(t))$ must stay the same for all large ts and thus $q(t)$ must also converge.

Now consider any 3×3 symmetric game that satisfies condition (B). Suppose there exists a t' such that $B(p(t')) = 1$. It follows from the monotone best-reply property that $B(\delta_1) = 1$. Since $p(t'+1)$ is a convex combination of $p(t')$ and $\delta_{B(\delta_{B(p(t'))})} = \delta_1$ one has $p(t'+1) \leq p(t')$. By the monotone best-reply property $B(p(t'+1)) \leq B(p(t'))$ and hence $B(p(t'+1)) = 1$. Continuing in this way it can be shown that $B(p(t)) = 1$ for all $t \geq t'$. Consequently both $p(t)$ and $q(t)$ will converge to δ_1. Similarly one can show that if $B(p(t')) = 3$ for some t' then $p(t)$ and $q(t)$ will converge to δ_3. Finally if $B(p(t)) = 2$ for all t then it is obvious that $q(t)$ will converge to δ_2 while $p(t)$ will converge to $\delta_{B(\delta_2)}$.

Q.E.D.

Proposition 3 suggests that, within the class of 3×3 symmetric games, learning process I may do just as well as fictitious play in providing a foundation for Nash equilibria. This raises the question of the merit of studying fictitious play. Why bother with fictitious play when there is an apparently better alternative learning process?

Notwithstanding the apparently strong convergence results given by propositions 1 and 3, learning process I has its own drawbacks that tends to discount its usefulness. First of all, beliefs generated by learning process I converge only to pure-strategy equilibria, if they converge at all. Thus a learning theory which is based on learning process I cannot justify the use of Nash equilibria in games with only mixed-strategy equilibria. Second and perhaps the most important, the learning process relies on fairly strong informational assumption which itself deserves a convincing explanation. In particular, why should one player be more informed and smarter than the other? Where does this player gets his information about his opponent's learning behaviour? Insofar as these questions are not satisfactorily addressed, fictitious play as a simple and yet useful learning process appears to deserve continual attention.

1.6 Suggestions for Future Research

The results presented in this chapter are confined to the class of 3×3 symmetric games. Some of them may carry on to games with more than three strategies, while others may require modifications or simply break down. I suggest some possible directions for future research.

(1). Theorem 1 states that for symmetric games with only three strategies, the existence of an asymmetric pure-strategy equilibrium will guarantee that fictitious play converges. Certainly this result does not hold for symmetric games with five or more strategies. For symmetric with n strategies, the sufficient condition may be the existence of m asymmetric pure-strategy equilibrium, where m is related to n in some special way.

(2). It would be useful to know if the monotone best-reply property alone is sufficient for fictitious play is sufficient to converge in symmetric games with more than three strategies.

(3). It would be interesting to characterize the limiting dynamic behaviour of beliefs generated by fictitious play in general two-person games. In particular one could address the question of whether beliefs would approach limit cycles when they do not converge to an equilibrium.

1.7 Conclusion

Recent attempts to provide a foundation for the concept of Nash equilibrium have concentrated on the limiting outcomes of learning behaviour in repeated strategic interaction. In particular, they look for clear and unambiguous answers to qestions such as: Can equilibrium analysis be justified by a learning theory? In what ways do the results depend on the particular learning process used by players? When learning does not converge to an equilibrium, what happens to the asymptotic behavior of players' beliefs?

I address the above questions in the context of 3×3 symmetric games with fictitious play as the relevant learning process. I identify a reasonably wide class of games in which beliefs generated by fictitious play would converge to the set of Nash equilibria. I also find that a pair of identical beliefs which does not converge to a symmetric Nash equilibrium will approach limit cycles. Finally I compare fictitious play with an alternative learning process that entails stronger informational and rationality requirements. Although the latter apparently provides stronger learning foundation for Nash equilibria, it suffers from the drawback that the source of additional information which is often so crucial to the results is left unexplained.

Although only a special class of two-person games is considered here, they have provided very useful insights into the dynamic implications of fictitious play. Certainly some of the results in this chapter do not carry on to games with more than three strategies. Nevertheless, they still suggest some directions for further research. It is hoped that the results obtained here would serve as useful references for future work in this area.

Footnotes

[1]There are many different learning models that vary in the way players are repeatedly matched to play the underlying game as well as the amount of information and sophiscation required of the specified learning process. For a comprehensive survey of learning models the reader may refer to Crawford (1993) and Mailath (1992).

[2]For two-player games, beliefs generated by fictitious play are consistent with a Bayesian learning model in which each player believes his opponent is playing some constant mixed strategy, and where each player's prior belief (which is a probability distribution over the space of mixed strategies available to his opponent) is given by a Dirichlet distribution. A more formal argument is given in Krishna (1992).

[3]convergence of beliefs or empirical distributions.

[4]This includes of course pure-strategy equilibrium.

[5]Given any symmetric game let $U(x, y)$ be the payoff of a player when he uses pure strategy x and his opponent uses pure strategy y. A game is said to satisfy strategic complementarity if the pure-strategy set can be ordered in such a way that for all $i \geq j$, $U(i, y) - U(j, y)$ is nondecreasing in y. Krishna (1992) shows that any game that satisfies strategic complementarity has the monotone best-reply property.

CHAPTER 2

GENERAL EQUILIBRIUM THEORY WITH MONOPOLISTIC COMPETITION: AN INTRODUCTORY ANALYSIS

2.1 Introduction

Ever since Leon Walras formulated the system of equations describing the general equilibrium of an economy, economists have been trying to know more about the properties of general economic equilibria. Following Arrow and Debreu's (1954) pioneer work on the existence of a general competitive equilibrium, general equilibrium theory has developed rapidly. An enormous body of literature is now in place which contains answers to almost every question that can be asked about the equilibrium of a perfect competitive economy.

Despite the rapid advancement of general equilibrium theory in the past three decades, existing equilibrium theory has very little to say about the equilibria of imperfect competitive economies. In fact, the major development of general equilibrium theory has focused almost exclusively on the study of perfect competitive equilibrium. Part of the reason for this bias is due to the analytical complexity introduced by non-price-taking behaviour. At the same time, formal equilibrium analysis of an imperfect competitive economy also faces the problem of choosing among the various possible models of imperfect competition.

The emphasis on the perfect competitive model would not be so much of a problem if it is a reasonably good approximation for a wide class of economies. If, however, there exist non-competitive economies with important features that are not satisfactorily captured by the former, then it seems necessary for us to model and study them in detail in order to gain a better understanding of how different economic systems work. This is particularly true when we believe that our own economic system is not adequately represented by a competitive model. Predictions and conclusions derived from the latter may be irrelevant and inapplicable to the former. Similarly, government policies that are appropriate for a competitive economy may be inappropriate for or even harmful to the real world economy.

Chapter 2 of my thesis is motivated by the belief that certain class of imperfect competitive economies do possess distinctive characteristics that are of interest from both theoretical and pragmatic points of view. Theoretically, the equilibria of these economies have properties that differ significantly from those of a competitive economy. Pragmatically, these properties give rise to certain policy implications that may be relevant to our own economy.

I develop and study a general equilibrium model of an economy in which some firms are capable of exercising monopoly or monopolistic power over the markets for their products. Each of these monopolistic firms perceives some subjective demand function for its product which satisfies a certain consistency condition in general equilibrium. The monopolistic firm chooses its level of output and sets the market price to maximize its expected profit, taking as given the prices of inputs. I define and prove the existence of monopolistic competitive equilibria as well as investigating the properties of these equilibria. Specifically, I show that under fairly standard assumptions, an infinite number of monopolistic competitive equilibria exists which are supported by different systems of perceived demand functions. If the cost functions of the monopolistic firms are sufficiently smooth, then the number of equilibria (with different prices and allocations) associated with a fixed system of perceived demand functions is generically finite. Also, different equilibria supported by different systems of demand functions are in general pareto incomparable.

It is only fair to mention that this is not the first general equilibrium model of monopolistic competition. Negishi (1961), Arrow (1971) and Mas-Colell (1982) have also studied the equilibrium problem of a monopolistic competitive economy.[1] The present study differs from the existing literature on monopolistic competitive equilibria in two novel ways. First, it goes beyond the question of existence of an equilibrium to the investigation of the properties of equilibria. Second, it employs weaker assumptions on the producers and the consumers than those made in the previous work.

The rest of chapter 2 is organized as follows. Section 2.2 describes the economy and the basic assumptions. Section 2.3 explains the problem of the monopolistic firms and gives a definition of a monopolistic competitive equilibrium. Section 2.4 proves the existence of an equilibrium and examines the question of the existence and uniqueness of equilibria with positive monopolistic production. Section 2.5 deals with the welfare properties of monopolistic competitive equilibria. Section 2.6 shows the generic finiteness of the number of equilibria associated with a given system of subjective demand functions. Section 2.7 discusses government policy implications of the results derived in the previous sections. Section 2.8 considers some extensions of the present model. Section 2.9 gives the summary and conclusion.

2.2 The Model

The model consists of an economy with m consumers, n producers and L goods. Without loss of generality assume that the first J goods are produced exclusively by J monopolistic firms. The remaining n - J firms behave competitively. Each monopolistic firm produces only one good and uses other goods as inputs. In addition, it takes as given the prices of its inputs and sets the price of its product to maximize its expected profit (a formal treatment of the monopolistic firm's decision problem will be given in section 2.3). All firms return their profits to the consumers through the latter's profit shares. All consumers behave competitively.

Let Y_i denotes the production possibility set of firm i and CO Y_i denotes the convex hull of Y_i. Furthermore, let $y_{i,-i} = (y_{i1}, y_{i2}, \ldots, y_{ii-1}, y_{ii+1}, \ldots, y_{iL})$ and $Y_i^-(y_{ii}) = \left\{ y_{i,-i} \in R_-^{L-1} \mid (y_{i1}, \ldots, y_{ii}, \ldots, y_{iL}) \in Y_i \right\}$. The following assumptions are made on the production sector of the economy.

(A1) $0 \in Y_i$ and Y_i is closed for all $i = 1, 2, \ldots, n$.

(A2) Y_i is convex for $J+1 \leq i \leq n$.

(A3) For $1 \leq i \leq J$, the input requirement set $Y_i^-(y_{ii})$ is convex for all $y_{ii} \geq 0$ and $Y_i^-(y_{ii}) \subseteq Y_i^-(y'_{ii})$ for all $y_{ii} \geq y'_{ii}$.

(A4) For $1 \leq i \leq J$, $Y_i^-(y_{ii})$ is a continuous correspondence at every

$y_{ii} \in$ proj Y_i, where proj Y_i is the projection of firm i's production possibility set on the i-th coordinate.

(A5) Let $\hat{Y} = \sum_{i=1}^{J} CO\ Y_i + \sum_{i=J+1}^{n} Y_i$, we have $R^l_- \subseteq \hat{Y}$ and $\hat{Y} \cap \{-\hat{Y}\} = \{0\}$.

Assumptions **(A1)** and **(A2)** are standard in general equilibrium analysis. Assumption **(A3)** and **(A4)** ensures that the set of cost-minimizing input combinations is convex and upper-hemi continuous. Assumption **(A5)** is similar to the one made by Arrow and Hahn (1971) in their study of a monopolistic competitive economy.

The following standard assumptions are made on the consumption sector of the economy.

(B1) The consumption set of consumer k or X_k is a closed and convex subset of R^l_+ for k = 1, 2,..., m.

(B2) Consumer preferences are continuous, strictly monotonic and convex.

(B3) consumer k's endowment vector ω_k is strictly positive.

Under assumptions **(A1)**, **(A2)**, **(A5)** and **(B1)** the set of attainable states of feasible allocations is bounded (see Debreu (1956)). Consequently, for equilibrium analysis it suffices to consider some compact subsets of Y_i and X_k. Let K be a compact cube of R^L that contains the feasible allocations in its interior and let $\hat{Y}_i = Y_i \cap K$, $\hat{X}_k = X_k \cap K$.

2.3 Monopolistic Competitive Equilibrium

Each monopolistic firm in the economy perceives some demand function that is linear in its own price.[2] Given the subjective demand function, the monopolistic firm chooses its own level of output and sets the market price to maximize its expected profit. Formally, the decision problem of firm j, $1 \le j \le J$, is given by[3]

$$\max_{p_j} \Pi_j(p_j, p_{-j}) = \max_{p_j} \left\{ p_j(a_j - b_j p_j)^+ + \sum_{i \neq j} p_i y_{ji} \right\} \tag{1}$$

where $a_j, b_j \ge 0$, $p_j(a_j - b_j p_j)^+ = \max \{ 0, p_j(a_j - b_j p_j) \}$ and $y_{ji} \le 0$ is the demand by firm j of input i.

Note that (1) can also be rewritten as

$$\max_{y_j} \Pi_j(y_j, p_{-j}) = \max_{y_j} \left\{ (a_j/b_j - 1/b_j y_{jj}) y_{jj} + \sum_{i \neq j} p_i y_{ji} \right\}$$

where y_j is chosen from the set $\hat{Y}_j$. As $\hat{Y}_j$ is compact the solution to the above problem exists. The optimal production correspondence $\hat{y}_j(p_{-j}, a_j, b_j)$ is upper-hemi continuous in (p_{-j}, a_j) given $b_j > 0$. Since the market price is a continuous function of output the optimal price correspondence $\hat{p}_j(p_{-j}, a_j, b_j)$ is also upper-hemi continuous in (p_{-j}, a_j). However, the fact that $\hat{Y}_j$ may not be convex implies that $\hat{y}_j(p_{-j}, a_j, b_j)$ (and hence $\hat{p}_j(p_{-j}, a_j, b_j)$) may not be convex-valued and this poses a problem in the latter proof of existence of an equilibrium. To eliminate this problem some restriction on the marginal cost function is needed.

Let $C_j(p_{-j}, y_{jj}) = \sum_{i \neq j} p_i \hat{y}_{ji}(p_i, y_{jj})$ be the cost function of firm j, where $\hat{y}_{ji}(p_i, y_{jj})$ is the cost-minimizing input demand on good i by firm j.

(A6) For $1 \le j \le J$, $C_j(p_{-j}, y_{jj})$ is continuously differentiable in y_{jj} for all $p_{-j} \ge 0$. In addition, $g_j(p_{-j}, y_{jj}) = \partial C_j(p_{-j}, y_{jj})/\partial y_{jj}$ is convex in y_{jj} and $\min\{\partial g_j(p_{-j}, y_{jj})/\partial y^+_{jj}, \partial g_j(p_{-j}, y_{jj})/\partial y^-_{jj}\} > -\delta$ for some $\delta > 0$.[4]

Lemma 1 shows that assumption **(A6)** guarantees unique profit-maximizing output and price provided the subjective demand functions are sufficiently inelastic. Define $C_j(p_{-j}, y_{jj}) = g_j(p_{-j}, y_{jj}) = \infty$ if y_{jj} cannot be feasibly produced given technology $\hat{Y}_j$, or more formally if $y_{jj} > \max\{\mathrm{proj}_j\ \hat{Y}_j\}$.

Lemma 1. Under assumption (A6), for all $b_j \le 2/\delta_j$, $\hat{y}_j(p_{-j}, a_j, b_j)$ and $\hat{p}_j(p_{-j}, a_j, b_j)$ contain only one element.

Proof: First note that $g_j(p_{-j}, y_{jj})$ is just the marginal cost function of firm j. Consider the following two cases.

Case (i) $g(p_{-j}, 0) \ge a_j/b_j$. In this case $g(p_{-j}, y_{jj}) > a_j/b_j - (2/b_j)y_{jj}$ for all $y_{jj} \in (0, a_j/2]$. Suppose not, then there exists some $y^*_{jj} \le a_j/2$ such that $g(p_{-j}, y^*_{jj}) \le a_j/b_j - (2/b_j)y^*_{jj}$. Note that

$$(g(p_{-j}, 0) - g(p_{-j}, y^*_{jj}))/y^*_{jj} \le -2/b_j \le -\delta.$$

By convexity of g_j one has

$$\partial g_j(p_{-j}, y_{jj})/\partial y^+_{jj} \le (g(p_{-j}, 0) - g(p_{-j}, y^*_{jj}))/y^*_{jj} \le -\delta,$$

which contradicts assumption (A6).

Since the marginal cost is everywhere greater than the marginal revenue the profit-maximizing output of firm j must be equal to zero. Therefore both $\hat{y}_j(p_{-j}, a_j, b_j)$ and $\hat{p}_j(p_{-j}, a_j, b_j)$ contain only one element.

Case (ii) $g(p_{-j}, 0) < a_j/b_j$. The marginal cost curve will intersect the marginal revenue curve at most once. Suppose not. Then there exists y^1_{jj} and y^2_{jj}, $y^1_{jj} \neq y^2_{jj}$, such that

$$g(p_{-j}, y^k_{jj}) = a_j/b_j - (2/b_j)y^k_{jj} \qquad k = 1, 2.$$

Without loss of generality let $y^1_{jj} < y^2_{jj}$. Note that

$$(g(p_{-j}, y^2_{jj}) - g(p_{-j}, y^1_{jj}))/(y^2_{jj} - y^1_{jj}) = -2/b_j \leq -\delta.$$

It follows by convexity of g_j that

$$\partial g_j(p_{-j},y_{jj})/\partial y^*_{jj} \leq (g(p_{-j}, y^2_{jj}) - g(p_{-j}, y^1_{jj}))/(y^2_{jj} - y^1_{jj}) \leq \delta.$$

which contradicts assumption (A6).

Now if $g(p_{-j}, y_{jj}) < a_j/b_j - (2/b_j)y_{jj}$ for every $y_{jj} \in \min\{\max(\mathrm{proj}_j\, \hat{Y}_j), a_j/2\}$ then $\hat{y}_j(p_{-j}, a_j, b_j) = \{\min\{\max(\mathrm{proj}_j\, \hat{Y}_j), a_j/2\}\}$, otherwise $\hat{y}_j(p_{-j}, a_j, b_j) = \{\bar{y}_{jj}\}$ where $\bar{y}_{jj}$ is given by $g(p_{-j}, \bar{y}_{jj}) = a_j/b_j - (2/b_j)\bar{y}_{jj}$. That $\hat{p}_j(p_{-j}, a_j, b_j)$ contains only one element follows immediately from the fact that there is a one-to-one relationship between $\hat{y}_j(p_{-j}, a_j, b_j)$ and $\hat{p}_j(p_{-j}, a_j, b_j)$.

Q.E.D.

For the competitive firm the profit maximizing problem is standard. The optimal production vector $\hat{y}_j$, $J + 1 \leq j \leq n$ exists for any given price vector p and the optimal production correspondence is convex-valued and upper-hemi continuous in p.

The decision problem of the consumer is also a standard one. Given a price vector p consumer k chooses a bundle $\hat{x}_k$ from the intersection of his consumption set $\hat{X}_k$ and his budget set

$$\beta_k(p, \omega_k, a, b) = \left\{ x_k \in R^l_+ \mid px_k \le p\omega_k + \sum_{j=1}^{J} \theta_{kj}[\hat{p}_j(a_j - b_j\hat{p}_j) - \sum_{i \ne j} p_i\hat{y}_{ji}] + \sum_{j=J+1}^{n} \theta_{kj}p\hat{y}_j \right\}$$

The demand correspondence of consumer k, denoted by $\hat{x}_k(p, a, b)$, is nonempty, convex-valued and upper-hemi continuous in (p, a) by assumptions (B2) and (B3) and the fact that the profit functions are continuous in (p, a) and $0 \in \hat{Y}_j$ for all j.

A definition of a monopolistic competitive equilibrium is now given as follows. A monopolistic competitive equilibrium is a $l + m + n + 2J$ tuple $(p^*, x^*, y^*, a^*, b^*)$ such that

(1) $x^*_k \in \hat{x}_k(p^*, a^*, b^*)$ for $1 \le k \le m$. That is, x^*_k maximizes consumer k's utility (or a maximal element on consumer k's budget set).

(2) y^*_j maximizes the profit of firm j given p^* for $J + 1 \le j \le n$ and (p^*_j, y^*_j) maximizes the profit of monopolistic firm j given (p^*_{-j}, a^*, b^*) for $1 \le j \le J$.

(3) Markets clear for all goods in the economy. That is,

$$\sum_{K=1}^{m} x^*_{ki} - \sum_{j=1}^{n} y^*_{ji} - \sum_{k=1}^{m} \omega_{ki} = 0 \qquad i = 1, 2, \ldots, L.$$

and

$$\sum_{k=1}^{m} x^*_{ki} - \sum_{j \neq i} y^*_{ji} - \sum_{k=1}^{m} \omega_{ki} = \begin{cases} a^*_i - b^*_i p^*_i, & a^*_i - b^*_i p^*_i \geq 0 \\ 0 & a^*_i - b^*_i p^*_i < 0 \end{cases} \qquad i = 1,\dots,J.$$

Note that condition (3) implies that the monopolistic firm may set a price higher than that necessary to drive market demand to zero. We like to point out that the present definition of a monopolistic competitive equilibrium is slightly different from that adopted by Negishi (1961) and Arrow and Hahn (1971) Neither of them allows an equilibrium price higher than that necessary to set demand for monopolistic output equal to zero.

2.4 Existence and Uniqueness of Monopolistic Competitive Equilibria

In this section I show that a monopolistic competitive equilibrium always exists for the economy under the assumptions made in section 2.3.

Theorem 1: Given assumptions **(A1)-(A6)**, **(B1)-(B3)**, there always exists a monopolistic competitive equilibrium with zero output by the monopolistic firms.

Proof: Consider the economy without the monopolistic firms. Then assumptions (A1), (A2), (A4) and (B1)-(B3) guarantee the existence of a competitive equilibrium (see Arrow and Hahn (1971); Debreu (1956); and Mas-Colell (1985)). Let the equilibrium price vector be denoted by p^*. Consider now the monopolistic firm j, let $b^*_j = 2/\delta$, $a^*_j = g_j(p^*_{-j}, 0)$ if

$p_j^* > g_j(p_{-j}^*, 0)$ and $a_j^* = p_j^*$ if $p_j^* \leq g_j(p_j^*, 0)$,and $y_j^* = 0$. Then the pair (p_j^*, y_j^*) maximizes firm j's profit for $1 \leq j \leq J$. By adding the tuple $((y_j^*), (a_j^*), (b_j^*))$ to the original competitive equilibrium it is straightforward to verify that we have a monopolistic competitive equilirium.

Q.E.D.

From the proof of Theorem 1 it is obvious that one can obtain multiple monopolistic competitive equilibria by choosing different values of a_j^*, given b_j^*, as long as the values of a_j^* are low enough that no monopolistic chooses to produce. These multiple equilibria are however uninteresting since they all have the same prices and consumption and production allocations.

Although Theorem 1 guarantees the existence of a monopolistic competitive equilibrium we are more interested in the question of existence of monopolistic competitive equilibria with positive production by at least some monopolistic firms. Unfortunately, the answer to this question is not necessary positive. In fact, it is easy to find example economies that satisfy assumptions (A1)-(A5) and (B1)-(B3) but which have no monopolistic competitive equilibria with positive monopolistic production. An example is given as follows.

Example 1. A monopolistic competitive economy with no equilibrium with positive output by some monopolistic firms.

Let $m = n = J = 1 = 2$. $U_1(x_{11}, x_{12}) = 2x_{11} + x_{12}$. $\theta_{11} = \theta_{12} = 1$

$$U_2(x_{21}, x_{22}) = x_{21} + 2x_{22},\ \theta_{21} = \theta_{22} = 0$$

and $\omega_{12} = 10$. $\omega_{12} = \omega_{21} = \omega_{22} = \varepsilon$, where ε is close to 0.

$$Y_1 = \left\{ (y_{11}, y_{12}) \in R^2 |\ y_{11} \geq 0,\ y_{11} = -0.5y_{12} \right\},$$

$$Y_2 = \left\{ (y_{21}, y_{22}) \in R^2 |\ y_{22} \geq 0,\ y_{22} = -y_{21} \right\},$$

where both firms are monopolistic.

The only equilibrium for this economy is the pure exchange equilibrium in which $p_1^* = 2p_2^*$. For if $p_1^* < 2p_2^*$ then consumer 1 will supply his endowment of commodity 2 for commodity 1. Firm 1 will not produce and whether firm 2 produces or not there will be an excess supply of good 2 in the economy. On the other hand, if $p_1^* > 2p_2^*$ then both consumers 1 and 2 will want to consume only commodity 2. As a result there will be excess supply o: ·ood 1. Note that even if one assume firm 2 behaves competitively there is still no monopolistic competitive equilibrium with positive production by firm 1.

The above example shows that it will be impossible to show the existence of monopolistic competitive equilibrium with positive monopolistic production generally. Nevertheless, one can still try to look for general answers to other closely related questions. Two such question will be: Is it possible to tell whether there exists a monopolistic competitive equilibrium with positive monopolistic production by looking at the set of competitive equilibria obtained by deleting the monopolistic competitive firms? Could different equilibria be pareto ranked?. Another equally important question is how many different equilibria could be supported by the same system of functions. I will consider each of these questions in turn.

For the rest of this chapter I will focus on an economy with only monopolistic firms since richer and more interesting results are obtained for such economies. The economy without the monopolistic firms is then equivalent to a pure exchange economy.

To answer the question concerning the relation of monopolistic competitive equilibria and pure exchange equilibria first note that for $J = I = n$ a strengthened version of the existence theorem is available.

Theorem 2. suppose $J = L = n$, then under assumptions (A1)-(A6), (B1)-(B5) , there exists, for each $b \leq 2/\delta$, an infinite number of monopolistic competitive equilibria such that

$$\sum_{k=1}^{m} x^*_{ki} + \sum_{i \neq j} y^*_{ji} + \sum_{k=1}^{m} \omega_{ki} = a^*_i - b^*_i p^*_i \qquad i = 1, 2, \ldots, L.$$

where $a^*_i, p^*_i > 0$.

Proof: Let $\phi_i(p, a) = \sum_{k=1}^{m} \hat{x}_{ki}(p, a) - \sum_{j \neq i} \hat{y}_{ji}(p_{-j}, a_j) - \sum_{k=1}^{m} \omega_{ki}$. Fix any $J\times 1$ vector $b \leq 2/\delta$ such that b_j represents the slope of firm j's subjective demand function. Choose a $J \times 1$ vector $\bar{a} > 0$ and let $S = \prod_{i=1}^{J} [0, \bar{a}_i]$, $A = \prod_{i=1}^{J} [0, \bar{a}_i/b_i]$. Define a correspondence

ρ: $S \times \bar{A} \longrightarrow \bar{A}$ by $\rho = \prod_{i=1}^{J} \rho$ where

$$\rho_i(p, a) = \begin{cases} \phi_i(p, a) \cap [0, \bar{a}_i] & \phi_i(p, a) \cap [0, \bar{a}_i] \neq \emptyset \\ \{ \bar{a}_i \} & \max\{ r: r \in \phi_i(p, a)\} \geq a_i \\ \{ 0 \} & \min\{ r: r \in \phi_i(p, a)\} \leq 0 \end{cases}$$

Here both consumers and producers are supposed to choose their consumption and production vectors from $\hat{X}_k$ and $\hat{Y}_j$ respectively. It is easy to verify that ρ_i is nonempty, convex-valued and upper-hemi continuous. In fact, let $z_{in} \in \rho_i(p^n, a^n)$ and suppose (p^n, a^n) converges to (p, a). If there exists an infinite subsequence of $\{z_{in}\}$, say, $\{z_{in(q)}\}$ such that $z_{in(q)} \in \phi_i(p^{n(q)}, a^{n(q)})$ for all $q \geq 1$, then it follows from the definition of ρ_i that $z_{in(q)} \in [0, \bar{a}_i]$ for all q. By the upper-hemi continuity of ϕ_i, $\{z_{in(q)}\}$ contains a subsequence which converges to an element in $\phi_i(p, a) \cap [0, \bar{a}_i]$ and hence to an element in $\rho_i(p, a)$. If $z_{in} \notin \phi_i(p^n, a^n)$ for all n large enough then one has either

$$\max\{r: r \in \phi_i(p^n, a^n)\} < 0 \quad \text{or} \quad \min\{r: r \in \phi_i(p^n, a^n)\} > a_i$$

for every large n. Therefore $\{z_{in}\}$ must contain a subsequence $\{z_{in(q)}\}$ such that $z_{in(q)} = 0$ for every q or $z_{in(q)} = \bar{a}_i$ for every q. Let

$$\bar{r}_{n(q)} = \max\{r: r \in \phi_i(p^{n(q)}, a^{n(q)})\},$$

$$r_{-n(q)} = \min\{r: r \in \phi_i(p^{n(q)}, a^{n(q)})\}.$$

Upper-hemi continuity of ϕ_i implies that $\{\bar{r}_{n(q)}\}$ (or $\{r_{-n(q)}\}$) contains a subsequence converging to some $\bar{r} \leq 0$ ($\underline{r} \geq a_i$) in $\phi_i(p, a)$. Since $\phi_i(p, a)$ is a convex set it follows that $0 \in \rho_i(p, a)$ ($a_i \in \rho_i(p, a)$). Hence the limit of $z_{in(q)}$ must belong to $\rho_i(p, a)$.

Define a continuous mapping T: $S \times \bar{A} \longrightarrow S$ by $T = \prod_{i=1}^{J} T_i$, where $T_i(p, a) = \rho_i(p_{-i}, a_i)$. Now define a correspondence ξ: $S \times \bar{A} \longrightarrow S \times \bar{A}$ by $\xi = T \times \rho$. The correspondence ξ is nonempty, compact-valued, convex-valued and upper-hemi continuous and goes from a compact convex set into itself. Therefore by Kakutani's fixed point theorem there exists a fixed point (p^*, a^*) such that $(p^*, a^*) \in \xi(p^*, a^*)$.

To show that $p^* > 0$ suppose $p_i^* = 0$ for some i, then it follows that $a_i^* = 0$ by virtue of the profit-maximizing hypothesis. But $p_i^* = 0$ implies that $z_i > 0$ for all $z_i \in \rho_i(p^*, a^*)$. This contradicts the hypothesis that $a^* \in \rho(p^*, a^*)$. Similar argument shows that $a^* > 0$.

Now consider a^*. If $a_i^* < \bar{a}_i$, then

$$a_i^* - b_i p_i^* = \sum_{k=1}^{m} \hat{x}_{ki}(p^*, a^*) - \sum_{j \neq i} \hat{y}_{ji}(p_{-j}^*, a_j^*) - \sum_{k=1}^{m} \omega_{ki}.$$

If $a_i^* = \bar{a}_i$, then

$$a_i^* - b_i p_i^* \leq \sum_{k=1}^{m} \hat{x}_{ki}(p^*, a^*) - \sum_{j \neq i} \hat{y}_{ji}(p_{-j}^*, a_j^*) - \sum_{k=1}^{m} \omega_{ki}.$$

If the equality holds, then market clears for good i. Otherwise there is excess demand. Suppose strict inequality holds for some good I, then one has

$$\sum_{k=1}^{m} \hat{x}_{kI}(P^*, a^*) - \sum_{j \neq I} \hat{y}_{jI}(p_{-j}^*, a_j^*) - \sum_{k=1}^{m} \omega_{kI} - a_I^* + b_I p_I^* > 0$$

and hence $p^*(\sum_{k=1}^{m} x_k^* - \sum_{j=1}^{J} y_j^* - \sum_{k=1}^{m} \omega_k) > 0$, where $x_k^* = \hat{x}_k(p^*, a^*)$ and $y_j^* = \hat{y}_j(p_{-j}^*, a_j^*)$. This however contradicts the Walras' law

$$p^*(\sum_{k=1}^{m} x_k - \sum_{j=1}^{J} y_j - \sum_{k=1}^{m} \omega_k) \leq 0.$$

Thus markets must clear for all goods.

It remains to show that x_k^* is a maximizer of consumer k's budget set corresponding to the entire consumption set X_k and (p_j^*, y_j^*) maximizes firm j's expected profit on the whole production set Y_j. Since all markets are clear the allocation $((x_k^*), (y_j^*))$ belongs to the set of attainable allocations. Therefore x_k^* lies in the interior of the compact convex cube K. If there exists a consumption bundle x_k in the intersection of the budget set $\beta_k(p^*, a^*)$ and X_k which is strictly better than x_k^*, then there must exist another bundle x_k' which is (i) a convex combination of x_k^* and $\bar{x}_k$; (ii) close enough to x_k^* such that it lies in $\hat{X}_k$ and $\beta_k(p^*, a^*)$; (iii) which is strictly preferred to x_k^* by assumption (B2). Now (ii) and (iii) contradicts the hypothesis that x_k^* is a maximizer on the set $\hat{X}_k \cap \beta_k(p^*, a^*)$.

Now consider any monopolistic firm j. If max $\text{proj}_j\ \hat{Y}_j$ = max $\text{proj}_j\ Y_j$ then obviously (y_j^*, p_j^*) maximizes firm j's expected profit on Y_j. Otherwise one has $y_{jj}^* <$ max $\text{proj}_j\ \hat{Y}_j$ since y_j^* is in the interior of K. As y_j^* maximizes firm j's profit on $\hat{Y}_j$ it follows from the proof of lemma 1 that $y_{jj}^* = 0$ or $g_j(p_{-j}^*, y_{jj}^*) = a_j^*/b_j - (2/b_j)y_{jj}^*$. In either case assumption (A6) implies

$$g_j(p_{-j}^*, y_{jj}) > a_j^*/b_j - (2/b_j)y_{jj}$$

for all $y_{jj} > y_{jj}^*$. Hence it would be unprofitable to expand output beyond y_{jj}^* even when firm j has access to the original technology Y_j. This proves the claim that (y_j^*, p_j^*) maximizes firm j's profit on Y_j.

To show that there exists an infinite number of equilibria suppose in contrary that only a finite number of equilibria exists. Let E be the set of equilibrium vectors a^*. Choose $\bar{\bar{a}} > 0$ such that $\bar{\bar{a}} < a^*$ for all $a^* \in E$ Letting $\bar{\bar{A}} = \prod_{i=1}^{J} [0, \bar{\bar{a}}_i]$ and $S = \prod_{i=1}^{J} [0, \bar{\bar{a}}_i/b_i]$ and repeating the same argument as above we will get another equilibrium vector (p'', a'') where $a'' \notin E$. This contradiction shows that the number of equilibria is infinite. The proof is complete.

Q.E.D.[5]

Although Theorem 2 states that there can be infinite number of equilibria for each given vector of slopes of the subjective demand function, one would still like to know how many of these equilibria are distinct in their equilibrium prices and/or allocations. It is not difficult to show that if two equilibria have different equilibrium intercepts for the subject demand curves then the equilibrium price vectors must be different. This result is given in corollary 1.

Corollary 1. Let (p^1, x^1, y^1, a^1, b) and (p^2, x^2, y^2, a^2, b) be two equilibria corresponding to Theorem 2. If $a^1 \neq a^2$ then $p^1 \neq p^2$.

Proof: Suppose $a^1 \neq a^2$ and $p^1 = p^2 = p$. Without loss of generality let $a^1_j > a^2_j$ and $y^i_j = a^i_j - b_j p_j$, $i = 1, 2$ for some j. Therefore $y^1_{jj} > y^2_{jj}$ if the market price is to remain the same. Now profit-maximization requires that $g_j(p_{-j}, y^1_{jj}) \leq a^1_{jj}/b_j - (2/b^1_j)y_{jj}$ and $g_j(p_{-j}, y^1_{jj}) >$ $a^2_j/b_j - (2/b_j)y^2_{jj}$. But

$$a^1_{jj}/b_j - (2/b_j)y^1_{jj} = 2p_j - a^1_j/b_j < a^2_j/b_j - (2/b_j)y^2_{jj} = 2p_j - a^2_j/b_j,$$

which is a contradiction. Hence $p^1 \neq p^2$.

Q.E.D.

Corollary 1 indicates that different beliefs may be used to support different equilibrium prices. However it says nothing about the equilibrium allocations. In fact, if $(p^*, x^*, 0, a^*, b^*)$ is one equilibrium, then $(tp^*, x^*, 0, ta^*, b^*)$ will also be an equilibrium for any arbitrary $t > 0$. With positive monopolistic production the economy will no longer be homogeneous of degree 0 in p and a. However, it will still be homogeneous of degree in p and 1/b. This means that if $(p^*, x^*, y^*, a^*, b^*)$ is an equilibrium $(tp^*, x^*, y^*, a^*, tb^*)$ will also be an equilibrium. Hence any two set of equilibria with different slope vectors are actually equivalent in terms of relative prices, allocations and intercepts of the subjective demand functions.

Corollary 2. Let $\xi(b) = \left\{ (p, x, y, a) \in S^l_+ \times R^{m+(n+1)l} \mid (kp, x, y, a, b) \right.$ is a monopolistic competitive equilibrium for some $\left. k > 0 \right\}$. Then $\xi(b) = \xi(\hat{b})$ for any b, $\hat{b} \neq 0$.

Proof. Let $(\hat{p}, \hat{x}, \hat{y}, \hat{a}) \in \xi(\hat{b})$. By definition there exists some $k > 0$ such that $(k\hat{p}, \hat{x}, \hat{y}, \hat{a}, \hat{b})$ is a monopolistic competitive equilibrium. Let $t = \hat{b}/b$ and consider any monopolistic firm j. I claim that

$$\Pi_j(tk\hat{p}_{-j},\ tk\hat{p}_j,\ \hat{a}_j,\ \hat{b}_j/t) \geq \Pi_j(tk\hat{p}_{-j},\ p_j^*,\ \hat{a}_j,\ \hat{b}_j/t)$$

for any $p_j^* \neq tkp_j$. In fact, suppose

$$\Pi_j(tk\hat{p}_{-j},\ tk\hat{p}_j,\ \hat{a}_j,\ \hat{b}_j/t) < \Pi_j(tk\hat{p}_{-j},\ p_j^*,\ \hat{a}_j,\ \hat{b}_j/t)$$

for some $p_j^* \neq tk\hat{p}_j$. Then since the cost function $C_j(p_{-j},\ a_j - b_jp_j)$ is homogeneous of degree one in p_{-j} has

$$\begin{aligned}\Pi_j(tk\hat{p}_{-j},\ tk\hat{p}_j,\ \hat{a}_j,\ \hat{b}_j/t) &= t\Pi_j(k\hat{p}_{-j},\ k\hat{p}_j,\ \hat{a}_j,\ \hat{b}_j) \\ &< \Pi_j(tk\hat{p}_{-j},\ p_j^*,\ \hat{a}_j,\ \hat{b}_j/t) \\ &= t\Pi_j(k\hat{p}_{-j},\ p_j^*/t,\ \hat{a}_j,\ \hat{b}_j)\end{aligned}$$

which implies that p_j does not maximize firm j's profit given $k\hat{p}_{-j}$ and its subjective demand function $\hat{a}_j - \hat{b}_jp_j$, a contradiction to the hypothesis that $(kt\hat{p},\ \hat{x},\ \hat{y},\ \hat{a},\ \hat{b})$ is an equilibrium.

Since $(tk\hat{p}_j,\ \hat{y}_j)$ maximizes firm j's profit, $1 \leq j \leq n$, and consumers' budget sets are homogeneous of degree 0 in the prices and firms' profits the tuple $(tk\hat{p},\ \hat{x},\ \hat{y},\ \hat{a},\ \hat{b}/t)$ naturally constitutes a monopolistic competitive equilibrium. Thus $(\hat{p},\ \hat{x},\ \hat{y},\ \hat{a}) \in \xi(b)$ and hence $\xi(\hat{b}) \subseteq \xi(b)$. By reversing the roles of $\hat{b}$ and b in the above argument one can show that $\xi(b) \subseteq \xi(\hat{b})$.

Q.E.D.

If the only equilibria satisfying the conditions of Theorem 2 are those with positive monopolistic production, then by Theorem 2 and corollary 1 one would obtain an infinite number of equilibria with either different allocations or prices or both. Proproposition 1 gives conditions under which one can tell whether multiple equilibria with positive monopolistic production exist.

Proposition 1. Let $J = L = n$ and suppose assumptions (A1)-(A6), (B1)-(B3) are satisfied. Consider the economy without the monopolistic firms. If for every exchange equilibrium there is a j^* such that $p^*_j >$ $g_j(p^*_{-j}, 0)$ where p^* is the equilibrium price vector, then there exists an infinite number of monopolistic competitive equilibria with positive monopolistic production. If in addition $\sup_y \{proj_j Y_j\} = \infty$ (i.e. every output can be feasibly produced given technology Y_j) then no two equilibria have the same relative prices and allocations.

Proof. I will show that if the condition of proposition 2 is satisfied, then no pure exchange equilibrium prices and allocation can be the prices and allocation of some monopolistic competitive equilibrium satisfying the conditions of Theorem 2.

Let (p^*, x^*) be a pure exchange equilibrium obtained by deleting the n monopolistic firms. Let $p^*_j > g_j(p^*_j, 0)$. Since $g_j(p^*_j, y_{jj})$ is continuous by assumption (A6), $g_j(p^*_{-j}, \varepsilon) < p^*_j$ for some $\varepsilon > 0$ Thus firm j^* will find it profitable to produce at least some positive quantity given any subjective demand function satisfying $a_j/b_j = p^*_j$ This in turn implies that no subjective demand function with $a_j - b_j$ $p^*_j = 0$ can be an equilibrium subjective demand function supporting the pure exchange allocation.

Thus no pure exchange equilibrium prices and allocation can be the prices and allocation of some monopolistic competitive equilibrium satisfying the conditions of Theorem 2. Hence from Theorem 2 there exists an infinite number of monopolistic competitive equilibria each with positive production by some monopolistic firms. Let (p^1, x^1, y^1, a^1, b) and (p^2, x^2, y^2, a^2, b) be two different equilibria. Consider the following two cases.

Case 1. $a^1 = a^2$. Suppose $p^1 = tp^2$, $t \neq 1$. With $a^1 = a^2$ the equilibrium output of every monopolistic firm at p^1 will be different from that at p^2 since demand curve remains the same but prices are different. Consequently $y^1 \neq y^2$ and the two equilibrium allocations must be different.

Case 2. $a^1 \neq a^2$. Suppose $p^1 = tp^2$ and $y^1 = y^2$. From corollary 1 $p^1 \neq p^2$ hence $t > 1$. Consider those firms with positive output at p^1. For those firms one has $a^1_j \neq a^2_j$ otherwise $y^1_j \neq y^2_j$. Since y^i_j $(i = 1, 2)$ is below the capacity output on $\hat{Y}_j$ lemma 1 implies that $g_j(p^i_{-j}, y^i_{jj}) = a^i_j/b_j - (2/b_j)y^i_{jj}$. Notice that by $g_j(p^1_{-j}, y^1_{jj}) = tg_j(p^2_{-j}, y^2_{jj})$ since cost-minimizing input demand is homogeneous of degree zero in input prices. Combining the two equations we have $a^1_j = ta^2_j - (t-1)2y^1_{jj}$. On the other hand, from $p^1 = tp^2$ and the inverse subjective demand functions one gets $a^1_j = ta^2_j - (t-1)y^1_{jj}$. Hence $y^1_{jj} = y^2_{jj} = 0$ which contradicts the hypothesis $y^1_{jj} = 0$.

Now suppose $p^1 = p^2$. As $a^1 \neq a^2$ there must be some firms that produce different output at p^1 and p^2. Thus the equilibrium allocations are different at p^1 and p^2.

$x^* = 4 - 0.25t,\ a^* = 1.5t,\ a^* = t,\ b^* = b^* = 1.$

Proposition 1 is important in the sense that it demonstrates how monopolistic competitive economies can differ significantly from pure exchange or perfect competitive economies. While one can guarantee the existence of at most finite number of equilibria almost everywhere (in a set theoretic sense) for the latter by imposing some smoothness assumptions on consumers' demand functions and the optimal production vectors (see Debreu (1970); and Mas-Colell (1985) for a comprehensive survey), the existence of an infinite number of monopolistic competitive equilibria with different relative prices and/or allocations is a possibility that may occur for a wide range of preferences and endowments, as long as the production technologies of monopolistic firms are efficient enough to ensure that no pure exchange equilibrium can be supported as a monopolistic competitive equilibrium satisfying the conditions of Theorem 2.

2.5 Welfare Properties of Monopolistic Competitive Equilibria

It is generally believed that imperfectly competitive markets will lead to inefficient allocations because market prices will exceed marginal costs of production. Such a belief stems from the partial equilibrium analysis given in standard microeconomic textbooks in which the monopolistic firms are supposed to perceive the correct demand functions. It is important to note that when the subjective demand functions do not coincide with the true ones but merely intersect the latter at the equilibrium prices and quantities the argument of inefficiency may not go through. The following example shows how an economy with monopolistic firms can actually lead to equilibrium allocations that are efficient.

Example 3. let $J = L = n = 2$, $m = 1$, $U_1(x_{11}, x_{12}) = 7x_{11} + x_{12}$, $\theta_{11} = \theta_{12} = 1$, $\omega_1 = (2, 4)$. Both firms are monopolistic competitive.

$$Y_1 = \left\{ (y_{11}, y_{12}) \in R^2 \mid y_{11}^2 = -y_{12},\ y_{11} \geq 0 \right\}$$

$$y_2 = \left\{ (y_{21}, y_{22}) \in R^2 \mid y_{22} = -0.25y_{21},\ y_{22} \geq 0 \right\}.$$

In this example, there is a monopolistic competitive equilibrium in which $p_1^* = 6$, $p_2^* = 1$, $y_{11}^* = 2$, $y_{12}^* = -4$, $y_{21}^* = = y_{22}^* = 0$, $a_1^* = 8$, $b_1^* = 1$, $a_2^* = 1$, $b_2^* = 1$, $x_1^* = (4, 0)$. If both firms now behave competitively (i.e. as price takers), it is easy to verify that the (p'', x^*, y^*) where $p_1'' = 4$, $p_2'' = 1$, constitutes a perfect competitive equilibrium. Since consumer's preference is strictly monotonic the competitive equilibrium is pareto optimal by the well known first welfare theorem. Thus the monopolistic competitive equilibrium, having the same allocation as the perfect competitive one, must also be pareto optimal.

The above example suggests that monopolistic competitive equilibria may be pareto efficient. Thus the usual criticism of inefficient monopolistic production may not be valid unless one is given detailed knowledge of the production technology of the monopolistic firms, their perceived demand functions and consumers' behaviour. However, proposition 1 indicates that there may be multiple monopolistic competitive equilibria and it is unlikely that all of them are pareto optimal. In general, when two equilibria are pareto suboptimal can something be said about their relative desirability? As the following proposition shows, if one only knows the equilibrium prices and production allocations it is in general unable to compare the two equilibria unless the two equilibrium price vectors satisfy some relationship and one monopolistic competitive equilibrium has a higher overall level of production than the other.

Proposition 2. Consider two monopolistic competitive equilibria $(p^1, x^1, y^1, a^1, b^1)$ and $(p^2, x^2, y^2, a^2, b^2)$. If consumer preferences are monotonic and (i) $p^1 = p^2$, $a^1 > a^2$ and $b^1 \leq b^2$ (or $a^1_j/b^1_j = a^2_j/b^2_j$) for all $j = 1, 2, .n$; or (ii) $p^1 = tp^2$, $a^1 = ta^2$ and $b^1 = b^2$, $t > 1$, then equilibrium 1 is strictly preferred to equilibrium 2. On the other hand, if $p^1 = tp^2$, $a^1 = a^2$ and $b^1 = b^2/t$, then the two equilibria are equivalent in terms of consumers' welfare.

Proof: (i) If $p^1 = p^2$, $a^1 > a^2$ and $b^1 = b^2$, then every monopolistic firm is producing more output at p^1 than at p^2. Consider any consumer k in the economy. His wealth consists of the value of his endowment and his profit share. Since $p^1 = p^2$ the value of his endowment remains the same in both equilibria. Profit for each monopolistic firm must be greater in equilibrium 1 than in equilibrium 2 since firm j can obviously produce y^2_j (the equilibrium output of firm j at equilibrium 2) and receives higher profit but it chooses to expand its output.

Consequently those consumers receiving a positive profit share from some monopolistic firms must have a greater wealth in equilibrium 1. As consumers preferences are strictly monotonic they must be strictly better off in equilibrium 1 than in equilibrium 2. Equilibrium 1 therefore is pareto superior to equilibrium 2.

For (ii), note that consumers budget constraints are homogeneous of degree 0 in prices and profits. In equilibrium 1 all prices are raised by a factor t, but profits are raised by a factor greater than t. Hence the net wealth of those consumers having positive profit shares of those firms producing positive output are better off in equilibrium 1 than in equilibrium 2. All other consumers' budget sets are essentially unaffected and hence their welfare does not change.

To prove the second part of the proposition note that theeconomy is homogeneous of degree 0 in (p, B), where $(B) = (1/b_1, \dots, 1/b_n)$. Thus if (p, x, y, a, b) is an equilibrium then (tp, x, y, a, b/t) will also be an equilibrium. As consumers' and producers' decisions are homogeneous of degree 0 in (p, B) $p^1 = tp^2$, $a^1 = a^2$ and $b^1 = tb^2$ implies that both consumers and producers remain the same on their welfare status in the two equilibria.

Q.E.D.

Example 4 gives a monopolistic economy with multiple equilibria that possesses the characteristics of (i) and (ii) of proposition 2.

Example 4. Let $J = m = n = L = 2$, $U_1(x_{11}, x_{12}) = 2x_{11} + x_{12}$, $U_2(x_{21}, x_{22}) = 2x_{21} + x_{22}$, $\theta_{11} = \theta_{12} = \theta_{21} = \theta_{22} = 1/2$, $\omega_1 = (6,6)$.

$\omega_2 = (3,3)$. The production function of firm 1 is $f_1(y_{12}) = -2y_{12}$ and that of firm 2 is $f_2(y_{21}) = -0.5y_{21}$.

For this economy there exists a continuum of monopolistic competitive equilibria with the same equilibrium prices. A general representation of the equilibria is given by

$p^* = (4, 2)$, $y^*_{12} = -(3/2)t$, $y^*_{21} = y^*_{22} = 0$, $x^*_1 = (6 + (3/2)t, 6 - (3/4)t)$, $x^*_2 = (3 + (3/2)t, 3 - (3/4)t)$, $a^*_1 = 7t$, $b^*_1 = t$, $a^*_2 = 2t$, $b^*_2 = t$, $0 < t \leq 4$.

In the above example, if b_j is fixed at 1 for $1 \leq j \leq J$, one may get another continuum of equilibria that can also be pareto ranked. These equilibria have the same relative prices and the higher the general price level, the better off the consumers. A general representation of the equilibria is given by

$p^* = (4t, 2t)$, $y^*_{11} = 3t$, $y^*_{12} = -(3/2)$, $y^*_{21} = y^*_{22} = 0$, $x^*_1 = (6 + (3/2)t, 6 - (3/4)t)$, $x^*_2 = (3 + (3/2)t, 3 - (3/4)t)$, $a^*_1 = 7t$, $a^*_2 = 2t$, $b^*_1 = 1$, $b^*_2 = 1$.

Note that this continuum of equilibria and the previous one have the same allocation representations.

If two monopolistic competitive equilibria do not satisfy the relationship specified in Proposition 2 then generally one cannot tell whether any one of them is better or they are pareto incompatible just by looking at their prices and information given by their equilibrium subjective demand functions. This can be seen from the following two examples.

Example 5. Let $n = J = L = 2$, $m = 1$, $U_1(x_{11}, x_{12}) = x_{11}x_{12}$, $\theta_{11} = 1$, $\theta_{12} = 1$, $\omega_{11} = 4$, $\omega_{12} = 2$.

$$Y_1 = \left\{ (y_{11}, y_{12}) \in R^2 \mid y_{21} \geq 0,\ y_{11} = -0.2y_{12} \right\}.$$

$$Y_2 = \left\{ (y_{21}, y_{22}) \in R^2 \mid y_{22} \geq 0,\ y_{22} = -5y_{21} \right\}.$$

For this economy there are two equilibria:

$$p_1^1 = p_2^1 = 1,\ a_1^1 = 3.125,\ a_2^1 = 5.625,\ b_1^1 = b_2^1 = 3.125$$
$$p_1^2 = 2,\ \ p_2^2 = 1,\ a_1^2 = 33.33,\ a_2^2 = 26.67,\ b_1^2 = b_2^2 = 16.67.$$

It is obvious that the consumer is better off in the second equilibrium than in the first equilibrium.

Example 6. Let $m = n = J = L = 2$, $U_1(x_{11}, x_{12}) = 2x_{11} + x_{12}$, $\theta_{11} = 1$, $\theta_{12} = 0$, $\omega_{11} = 3$, $\omega_{12} = 6$, $U_2(x_{21}, x_{22}) = x_{21} + x_{22}$, $\theta_{21} = 0$, $\theta_{22} = 1$, $\omega_{21} = 2$, $\omega_{22} = 4$,

$$Y_1 = \left\{ (y_{11}, y_{12}) \in R^2 \mid y_{11} \geq 0,\ y_{11} = -2y_{12} \right\}.$$

$$Y_2 = \left\{ (y_{21}, y_{22}) \in R^2 \mid y_{22} \geq 0,\ y_{22} = -0.5y_{21} \right\}$$

There are two equilibria for this economy, which are given by

$p_1^1 = p_2^1 = 1$, $a_1^1 = 36$, $a_2^1 = 24$, $b_1^1 = b_2^1 = 24$ and $p_1^2 = 2$, $p_2^2 = 1$, $a_1^2 = 28/3$, $a_2^2 = 14/3$, $b_1^2 = b_2^2 = 14/3$.

One can easily check that consumer 1 is better off and consumer 2 worst off in equilibrium 1 than in equilibrium 2.

The ambiguity in the relative welfare of equilibria with different relative prices may be explained intuitively as follows. If the relative prices of two equilibria differ, then some commodities must be more expensive and other commodities less expensive in one equilibria than in the other. The effects of a change in relative prices on the typical consumer in the economy depends on his preferences, the distributions of his endowed commodities and his profit shares in the firms. If he has a relatively abundant endowment of those commodities whose prices have gone up and a large profit share in those firms whose profits have increased, his wealth will increase and consequently he must be better off.

On the other hand, if his endowment mostly concentrates on those commodities whose prices have fallen and he has a strong preference for those commodities which are now more expensive, plus he has a low profit shares, then it follows that his welfare must deteriorate. The relative welfare of the two equilibria therefore depends on the those characteristics as the distribution of preferences, endowments and profit shares as well as the conjectures of the firms in the two equilibria (for example, in the first example above, although the input price is higher in the second equilibrium, firm 1 expects higher demand in the second equilibrium and therefore expands its production. This leads to higher profit and through the profit share the wealth and hence the welfare of the consumer is increased.).

2.6 Finiteness of Monopolistic Competitive Equilibria Supported by a Fixed System of Subjective Demand Functions.

I now turn to the question of how many monopolistic competitive equilibria can be supported by a given system of subjective demand functions. Although Proposition 1 indicates that there may exist an infinite number of different equilibria for a monopolistic competitive economy, such an infiniteness result does not seriously undermine the usefulness of the general equilibrium monopolistic competitive model since one can explain the emergence of different equilibria by the different beliefs held by firms. A more serious problem, however, is the existence of an infinite number of equilibria that are supported by the same beliefs. In this case it becomes very difficult to explain why a particular equilibrium associated with a given system of subjective demand functions emerges.

Can one ensure the existence of at most finite number of equilibria that are supported by the same system of subjective demand functions? It turns out that if the cost functions of the monopolistic firms are sufficiently smooth and consumers preferences are strictly convex then one would be able to guarantee the existence of only a finite number of equilibria for almost every system of subjective demand functions. In what follows the term equilibrium referes to one that satisfies the conditions in Thereom 2. That is, in equilibrium no monopolistic firm is allowed to set a price higher than that necessary to drive market demand to zero.

The additional assumptions are stated as follows:

(B2') Consumer preferences are continuous and strictly convex.

(A6') $C_j(p_{-j}, y_{jj})$ has continuous second order partial derivatives in all its varibles. Furthermore, for all $p_j \geq 0$ and $y_{jj} \in \text{proj}_j Y_j$,

$g_j(p_{-j}, y_{jj}) = \partial C_j(p_{-j}, y_{jj})/\partial y_{jj}$ is convex in y_{jj} and $\partial g_j(p_{-j}, y_{jj})/\partial y_{jj} = \partial^2 C_j(p_{-j}, y_{jj})/\partial y_{jj}^2 > -\delta$ for some $\delta > 0$.

Assumptions **(A6')** and **(B2')** actually imply that for each equilibrium price vector and system of subjective demand functions, there corresponds an unique equilibrium consumption and production allocation. To show the generic finiteness of monopolistc competitive equilibria it then suffices to show the existence of a finite number of equilibrium price vectors almost everywhere.

Consider any open set $U = (\underline{a}, \bar{a}) \subseteq R^n_+$. By corollary 2 one needs only to consider the set of equilibria corresponding to a fixed slope vector $0 < b \leq 1/\delta$, where δ is specified in assumption (A6). For each (a, b), $a \in U$, one can define a vector of linear subjective demand functions such that (a_j, b_j) $(j = 1,2,....n)$ gives the intercept and the slope of firm j's subjective demand function. Thus $U \times \{ b \}$ corresponds to a subset of collections of subjective demand functions. Given any $(a, b) \in U \times \{ b \}$ denote the set of monopolistic competitive equilibria by $\xi(a)$ and the set of equilibrium price vectors by $\xi_p(a)$.

Next consider the price game played by the n monopolistic firms where the payoff function and the strategy set of each firm are given by its profit function $\Pi_j(p_j, p_{-j}, a_j, b_j)$ and the set $A_j = [0, \bar{a}_j/\underline{b}_j]$ $(\underline{b} > 0)$ respectively. In this game firm j chooses a price p_j from A_j to maximize its expected profit or payoff given all other prices and its own subjective demand function. Denote the set of Nash equilibria of this game by $\eta(a, b)$.

It follows immediately from the definition of a monopolistic competitive equilibrium (see section 3 above) that $\xi_p(a) \subseteq \eta(a, b)$. Thus if one can show that $\eta(a, b)$ is finite for almost every (a, b) the generic finiteness of $\xi(a)$ would then follow from the one-to-one correspondence between $\xi_p(a)$ and $\xi(a, b)$. Proposition 3 shows that $\eta(a, b)$ is actually finite for any (a, b) belonging to a dense open subset of U.

Proposition 3. Under assumption (A6') there exists an open dense subset E of U such that for any $(a, b) \in E \times \{ b \}$, $\eta(a, b)$ is finite.

Proof: Define a mapping $\psi : U \times R^n_+ \to R^n_+$ by $\psi_j(a, p) = \partial\Pi_j(p_j, p_j, a_j, b_j)/\partial p_j$, $j = 1, 2, \ldots, n$. Consider the derivative matrix of ψ:

$$\begin{vmatrix} \partial\psi_1/\partial a_1, \ldots, \partial\psi_n/\partial a_n & \\ \vdots & \\ \partial\psi_k/\partial a_1, \ldots, \partial\psi_n/\partial a_n & \partial\psi/\partial p' \\ \vdots & \\ \partial\psi_n/\partial a_1, \ldots, \partial\psi_n/\partial a_n & \end{vmatrix} = \begin{vmatrix} \lambda_{11}, 0, \ldots \ldots, 0 & \\ \vdots \qquad \vdots & \\ 0 \ldots, \lambda_{kk}, \ldots, 0 & \partial\psi/\partial p' \\ \vdots \qquad \vdots & \\ 0 \ldots, 0 \ldots, \lambda_{nn} & \end{vmatrix}$$

where $\lambda_{kk} = 1 + b_k \partial g_k(p_{-k}, y_{kk})/\partial y_{kk} \neq 0$ by assumption (A6). Clearly the derivative matrix has rank n at any $(a, p) \in U \times R^n_+$. Therefore ψ must be transversal to the zero dimensional manifold $\{ 0 \} \subset R^n_+$. By the theorem of transversality (see Mas-Colell (1985)) there exists a dense, open subset $E \subseteq U$ such that for every $a \in E$, the mapping ${}_a\psi_b : R^n_+ \longrightarrow R^n_+$ defined by ${}_a\psi_b(p) = \psi(a, b, p)$ is transversal to $\{ 0 \}$. An application of the inverse function theorem then implies ${}_a\psi_b^{-1}(0) = \eta(a, b)$ must be a zero-dimensional manifold.

Now for any given (a, b) the feasible choice set for each monopolistic firm must be bounded if it is not allowed to set a price higher than necessary to drive perceived market demand to zero. Thus ${}_a\psi_b^{-1}(0) = \eta(a, b)$ must be a bounded set. This, together with the fact that $\eta(a, b)$ is also a manifold of zero dimension, implies that $\eta(a, b)$ must be a finite set.

Q.E.D.

Proposition 3 combined with assumption **(B2')** gives the generic finiteness of $\xi(a,b)$

Theorem 3. Under assumptions **(A6')** and **(B2')**, there exists an open dense subset E of U such that $\xi(a)$ is finite for any $a \in E$.

Proof: This follows immdiately from proposition 3 and the fact that $\xi_p(a) \subseteq \eta(a, b)$.

Q.E.D.

2.7 Discussion

In this section I discuss some policy implications of my results and consider some natural extensions of the present model.

The existence of multiple equilibria in a monopolistic competitive economy that are supported by different subjective demand functions indicates that in such an economy firms' beliefs play an important role in determining the equilibrium that would prevail. Unlike in a perfect competitive economy where firms take the prices as given and presume they can sell whatever quantity of output they produce, here the monopolistic firms make their production decisions on the basis of the information they have on the market condition. If firms believe that they are constrained in their sales or demand for their products is low, they may produce less output together. This would result in lower aggregate income and hence lower levels of consumptions by consumers. Lower output by the firms also implies lower intermediate demand for goods. Consequently pessimistic beliefs may become self-fulfilling and the economy is stuck at low a level of activity. By the same reasoning, optimistic beliefs may be self-fulfilling and give rise to a blooming economic state.

The possibility of having an undesirable economic state because of pessimistic beliefs held by the firms suggests that there may be some room for government intervention. However, unlike the conventional argument for government intervention that centers around the role of the government to correct distortions introduced by imperfect competition (such as higher prices than marginal costs), here the government is called for to induce an optimistic outlook among the firms if an increase in general output is desirable.

To accomplish this goal the government may try to create an atmosphere of economic expansion through expansionary fiscal or/and monetary policies. If the firms believe that such policies are going to improve the economic situation they may react by expanding their output and thereby raising the aggregate economic activity. An alternative and more direct solution is to let the government buy goods (which may be financed by imposing lump sum taxes on the consumers) from the firms and redistribute them to the consumers through proportional transfers. However, the implementation of such a transfer program may be very costly and may offset some of the benefits of a higher level of aggregate output. Also, it tends to make the firms overdependent on the government such that termination of the program is likely to result in recession.

Another solution may be through nationalization of those key industries which require large quantities of basic inputs (such as labor and electricity) in production. Examples of these are the automobile industry and transportation industry. During those adjustment periods when the economy is hit by some adverse shocks the monopolistic firms may easily become pessimistic. As a result not only the effects of the shocks are aggravated but even after the shock has gone the economy cannot return to its former state since pessimistic beliefs are self-fulfilling. If those firms are run by the government the negative effects of the adverse shocks could be kept to a minimum and the economy could also recover more quickly through the maintenance of an optimistic outlook despite short term losses.

Nationalization of private firms may be done for a short period of time when the economy is affected by negative shocks and the private owners are too pessimistic to run their firms. After the general economic condition has improved the government can return them to the private sector. Prolonged nationalization may not be desirable given the fact inefficient management is usually involved in public enterprises.

2.8 Extensions of the Basic Model

I briefly discuss some possible extensions of my model and potential areas for further research.

(1) Stronger consistency for equilibrium subjective demand functions. The monopolistic competitive equilibrium defined here does not require the subjective demand functions to have the correct functional forms or even the correct slopes at the equilibrium prices. While it may be reasonable to assume that firms employ linear approximations to their subjective demand functions, it may be less convincing to assume that firms are satisfied to know that their prediction of sales is correct at the observed prices.

In the real world monopolistic firms are constantly updating their information on the market condition. One important piece of information they are looking for is the elasticity of the demand for their products. If they find out that they initial assessment of the elasticity is incorrect they may revise their beliefs and change the prices of their products accordingly. If one takes into account this fact then the definition of a monopolistic competitive equilibrium should be modified to include the condition that the subjective demand functions should have the correct slopes.

To prove the existence of the newly defined monopolistic competitive equilibrium we may have to impose more assumptions on the consumers and the producers. Indeed, if one assumes that the consumer demand functions and firms' input demand functions are continuously differentiable and that there exists a bounded set D such that for all $b \in D$,

$$\sum_{k=1}^{m} \partial \hat{x}_{ki}(p, a, b)/\partial p_i - \sum_{j \neq i} \partial \check{y}_{ii}(p_{-j}, a_j, b_j)/\partial p_i \in D_i, \text{ where } D_i$$

is the projection of D on the i-axis, then one can show, by following similar arguments as those given in the proof of Theorem 2, that there exists an infinite number of monopolistic competitive equilibria in which the subjective demand functions have consistent slopes.

The question of existence of an equilibrium then boils down to the question of existence of the set D. One may try to specify conditions under which D exists, or otherwise show that it is impossible to find such a set D under general assumptions on the economy. There may be alternative ways of showing the existence of an equilibrium with consistent slopes for the subjective demand functions, but they may also involve substantial modifications of the proof outlined above.

(2) Uncertainty. Throughout this paper it is assumed that each monopolistic firm perceives only one single subjective demand function for its product. More generally, the monopolistic firms should be allowed to work with some probability distributions on a space of demand functions (In the present model this corresponds to each monopolistic firm assigning probability one to some specific demand function.). If one considers only linear demand functions this would

mean that each monopolistic firm j perceives a probability distribution function υ_j on the intercept and slope of its subjective demand function. The monopolistic firm then chooses its price and output to maximize its expected profit

$$E(\Pi_j(\upsilon_j, p_j, y_{jj})) = p_j E[\min \{ a_j - b_j p_j, y_{jj} \}] - C_j(p_{-j}, y_{jj})$$

(assuming that whenever supply and demand differ at the chosen price, the short side of the market gives the sales of the monopolistic firm).

With this generalized formulation an equilibrium (which I shall refer to as the generalized equilibrium) is defined as a tuple (p^*, y^*, x^*, υ^*) such that x^*_k maximizes consumer k's utility, (p^*_j, y^*_{jj}) maximizes the expected profit of firm j given p^*_{-j} and υ^*_j ($\upsilon^*_j(A) = 0$ where A is defined by the following property; $(a_j, b_j) \in A$ implies $a_j - b_j p_j \neq y^*_{jj}$) and all the markets clear. Such an equilibrium must exist since the equilibria we found in section 4 can be reinterpreted as generalized equilibria in which $\upsilon^*_j(a^*_j, b^*_j) = 1$.

The generalized model may enable us to handle a stochastic environment in which consumers's preferences, endowments and production technology are dependent on the states of nature. In this case the probability distribution on subjective demand functions may reflect the information the monopolistic firm has on the state on nature. In particular, one can define fully revealing, partially revealing and non-revealing rational expectations equilibria in the spirit of Radner (1979). The study of the existence of such equilibria may represent an interesting area of research.

(3) Stability. An important topic that has not been investigated is the stability of monopolistic competitive equilibria. The issue of stability is much more complicated in a monopolistic competitive economy than in a perfect competitive economy. When a monopolistic competitive economy is in disequilibrium, it implies that some monopolistic firms have incorrectly predicted demand for their products at the prevailing market prices. Given the market information, those monopolistic firms must revise their subjective demand curves. They may adjust the intercepts, the slopes, or both as well as the prices of their products. If the revised set of prices and subjective demand function still do not constitute an equilibrium, some firms (or some other firms) will find it necessary to adjust again.

Disequilibrium adjustment in a monopolistic competitive economy therefore involves learning on the part of the monopolistic firms. Whether the adjustment process will converge to an equilibrium is a question that deserves careful study. It is possible that under very reasonable assumptions on the economy the adjustment process does not converge. One may approach the problem by modeling the adjustment process as a learning process in a dynamic game context. There is already a sizable literature on the subject of learning (e.g. see chapter 1) in games. The application of results from this literature may provide some answers to the stability problem discussed here.

(4) Infinite number of monopolistic firms. The model here has only a finite number of monopolistic firms. A more natural framework to analyze monopolistic competition would be one which allows for an

infinite number of differentiated products each of which is produced by a single monopolistic firms. Such a model has been presented and studied by Hart (1985), although his model is not entirely in the spirit of general equilibrium theory. General equilibrium with an infinite number of commodities have been studied in considerable detail for the case of a competitive economy. The results from the study of such an economy may provide useful hints to the equilibrium problem in a monopolistic competitive economy.

2.9 Summary and Conclusion

I have constructed and studied a general equilibrium model of a monopolistic competitive economy and studied the properties of its equilibria. The results derived from the analysis of the model show that in a monopolistic competitive economy there typically exists an infinite number of equilibria. Different beliefs held by the monopolistic firms may give rise to different equilibria some of which may be pareto ranked. Almost everywhere, however, the same beliefs can only support a finite number of equilibria.

It is evident that very different results could be obtained once one departs from the competitive model. By studying non-competitive but more realistic economics, one may gain valuable insights into the workings of the real economy. Imperfect competitive economies have received increasing attention and have been carefully studied in the macroeconomic literature (for example, see Hart (1982), Blanchard and Kiyotaki (1987) and Cooper and John (1988) for a discussion of imperfect competitive economies and multiple equilibria. To some extent, my results complement and generalize the results of these authors.), but they have not been rigorously studied by the general equilibrium theory. Hopefully this paper would provide a starting point and inspire more contribution to the general equilibrium theory of imperfect competitive economies.

Footnotes

[1]There is a very large volume of literature on monopolistic competition (see for example, Hart (1985), Dixit and Stiglitz (1977), and Perloff and Salop (1983) for a good presentation of monopolistic competitive models.). It is important to note that the models studied in this literature are not truly general equilibrium models in the sense that they restrict consumers' preferences to be symmetric, do not consider intermediate transactions between firms, and/or do not consider redistribution of profits of the firms to the consumers. Most important of all, they do not address questions that general equilibrium theory seeks to answer.

[2]Negishi (1961) assumes a subjective inverse demand function of the following form

$$p_j = f(y_{jj}, p_{-j}, \alpha_j, \theta_j)$$

for monopolistic firm j, where $(p_{-j}, \alpha_j, \theta_j)$ is a vector of parameters representing the general condition of the economy, and $f(\theta_j, p_{-j}, \alpha_j, \theta_j) = \alpha_j$. By assuming that the profit function is concave in y_j he is able to prove the existence of a monopolistic competitive equilibrium. The linear subjective demand functions assumed in the present study can be viewed as a special case of the Negishi subjective inverse demand functions where $a_j = (\theta_j k(p_{-j}) + \alpha_j)/k(p_{-j})$ and $b_j = k(p_{-j})$ (here $k(p_{-j})$ is any continuous function of p_{-j}). The assumption of linearity makes possible the explicit treatment of many important problems other than the existence of an equilibrium. As one can see in subsequent sections, many interesting and important results can be derived from linear model.

[3]Here the subjective demand functions are assumed to be independent of all other variables in the economy to avoid unnecessary complications in the presentation. All of our results continue to hold if we replace a_j and b_j by $a_j(p_{-j}, \sum_{k=1}^{m} x_{kr} - \sum_{j \neq i} y_{jr}, r \neq j)$ and $b_j(p_{-j}, \sum_{k=1}^{m} x_{kr} - \sum_{j \neq i} y_{jr}, r \neq j)$, as long as we require $a_j(p^*_{-j}, \sum_{k=1}^{m} x^*_{kr} - \sum_{j \neq i} y^*_{jr}) = a^*_j$, and $b_j(p^*_{-j}, \sum_{k=1}^{m} x^*_{kr} - \sum_{j \neq i} y^*_{jr}) = b^*_j$, where a * denotes an equilibrium (see section 3 in the text).

[4]$\partial g_j(p_{-j}, y_{jj})/\partial y^+_{jj}$ and $\partial g_j(p_{-j}, y_{jj})/\partial y^-_{jj}$ represent the left-hand and right-hand partial derivatives of g_j with respect to y_{jj}. Both exist by assumption of convexity of g_j.

[5]We could as well have proved the theorem using a game theoretic approach. We give a brief description of how it works. Let the monopolistic competitive economy be represented by an abstract economy with m+n+1 players. The first m players corresponds to the consumers and the next n players to the monopolistic firms. The last player is an auxiliary player who chooses the subjective demand functions for the monopolistic firms. The strategy sets of the first m players correspond to the consumers' budget sets. The strategy of monopolistic firm j is a pair $(p_j, y_j) \in [0, a_i/b_j] \times \hat{Y}_\ell$. The auxiliary player chooses a vector of intercepts **a** from the set $\prod_{i=1} [0, \bar{a}_i]$.

Payoffs to the first m + n players are given by the consumers' utilities and the firms' profits. The payoff function of the auxiliary player is given by

$$A(p, x, y, a) = \sum_{i=1}^{L} -(a_i - \sum_{k=1}^{m} x_{ki} - \sum_{j \neq i} y_{ji} - \sum_{k=1}^{m} \omega_{ki})^2$$

It is easy to see that the optimal strategy of the auxilary player is unique (given (p, x, y)) and is a continuous function of (p, x, y).

Given any $b > 0$, the abstract economy described above satisfies all the conditions outlined in Arrow and Debreu (1954) and hence it has an equilibrium. By applying the arguments in the later part of the proof of Theorem 2 one can easily verify that the equilibrium is a monopolistic competitive equilibrium.

[6]In fact, if it is assumed that max $\{\sup \{ x_i \mid x_i \in \hat{x}_{ki}(p, a)\}\} \longrightarrow \infty$ for $p \longrightarrow \bar{p} \in \partial S^l_l$, where S^l_l the unit simplex, ∂S^l_l the boundary of S^l_l and $p \in S^l_l$, for some k, then it can be shown that there exists a $\varepsilon(\omega) > 0$ such that $g_j(p_{-j}, 0) < \varepsilon(\omega)$ for all $p_{-j} \geq 0$ implies that there exist an infinite number of monopolistic competitive equilibria with positive output by some monopolistic firms.

APPENDIX: Proof of Lemmas and Theorems in Chapter 1.

Proof of lemma 3: Consider the two boundaries $\partial(B(p(t)),j)$ and $\partial(B(\delta_{B(q(t))},j))$. Since there does not exist a $\bar{p}$ such that $BR(\bar{p}) = \{1, 2, 3\}$ we have $\partial(B(p(t)),j) \cap \partial(B(\delta_{B(q(t))},j)) = \emptyset$. Let $\partial(B(p(t)),j) = \partial_1$ and $\partial(B(\delta_{B(q(t))},j)) = \partial_2$. Define

$$d(\partial_1, \partial_2) = \inf_{p\in\partial_1} \{\inf_{p\in\partial_2} \|p - q\|\}$$

where $\|p - q\|$ is the Euclidean distance between p and q. $d(\partial_1, \partial_2)$ is thus the shortest distance between two bounded line segments in S_1^1. Note that $d(\partial_1, \partial_2) > 0$, for otherwise there would exist a sequence $\{p_n\}_{n=1}^{\infty} \subseteq \partial_1$ with $\inf_{q\in\partial_2} d(p_n, q) < 1/n$ which, by compactness of ∂_1, contains a convergent subsequence converging to a point $p' \in \partial_1$. But $\inf_{q\in\partial_2} \|p' - q\| = 0$ and hence $p' \in \partial_2$, which contradicts our hypothesis.

Now set T such that $1/T < d(\partial_1, \partial_2)$ and consider any $t \geq T$. if $[p(t), \delta_{B(q(t))}] \cap B^{-1}(j) \neq \emptyset$ we can find a $p_0 \in [p(t), \delta_{B(q(t))}]$ such that $B(p_0) = j$. Note that $[p(t), p_0] \cap B^{-1}(B(\delta_{B(q(t))}) = \emptyset$ by convexity of $B^{-1}(B(\delta_{B(q(t))})$. Represents a typical point in $[p(t), p_0]$ by $p_0^t(\alpha) = \alpha p_0 + (1-\alpha)p(t)$, $0 \leq \alpha \leq 1$. We have $p_0^t(0) = p(t)$ and $p_0^t(1) = p_0$. Let $V(i, p)$ be the expected payoff of player 1 when his belief is p and he uses pure strategy i, i = 1, 2, 3. Now $V(B(p(t), p_0^t(\alpha)) - V(j, p_0^t(\alpha))$ is linear in α and can be viewed as a continuous function of α. Moreover, we have

$$V(B(p(t)), p_0^t(0)) - V(j, p_0^t(0)) \geq 0$$

and

$$V(B(p(t)),\ p_0^t(1)) - V(j,\ p_0^t(1)) \leq 0.$$

Consequently there exists an $\alpha_1 \in [0, 1]$ such that

$$V(B(p(t)),\ p_t^0(\alpha_1)) - V(j,\ p_0^t(\alpha_1)) = 0$$

and therefore $BR(p_0^t(\alpha_1)) = \{B(p(t)), j\}$.

Next consider the interval $[p_0, \delta_{B(q(t))}]$. By lemma 1 we have $[p_0, \delta_{B(q(t))}] \cap B^{-1}(B(p(t)) = \emptyset$. Repeating the above arguments we can find a unique $p'' \in [p_0, \delta_{B(q(t))}]$ such that $BR(p'') = \{j, B(\delta_{B(q(t))})\}$. Note that $p_0^t(\alpha_1) \in \partial_1$ and $p'' \in \partial_2$ and therefore

$$\|p_0^t(\alpha_1) - p''\| \geq d(\partial_1, \partial_2) > 0.$$

By hypothesis of the lemma player 2 will continue to use the strategy $B(q(t))$ as long as player 1 stays with strategy $B(p(t))$ (starting from period t). Let t' be the first time player 1 changes his strategy. That is, $B(p(t+k)) = B(p(t))$ for $0 \leq k \leq t' - t - 1$ and $B(p(t')) \neq B(p(t))$.

Now fictitious play implies that $\|p(t') - p(t'-1)\| \leq 1/t'$. Since $t' > T$ we have

$$\|p(t') - p(t'-1)\| \leq 1/t' < 1/T < d(\partial_1, \partial_2).$$

If $B(p(t')) = B(\delta_{B(q(t))})$ we have $p(t') \in [p'', \delta_{B(q(t))}]$ and therefore

$$\|p(t') - p(t'-1)\| > \|p_0^t(\alpha_1) - p''\| \geq d(\partial_1, \partial_2),$$

which is a contradiction. Hence $B(p(t')) = j$.

Q.E.D.

Proof of lemma 4: The proof is very similar to that of lemma 3, except in this case we consider the sets $\partial_1 \backslash U_\varepsilon(\bar{p})$ and $\partial_2 \backslash U_\varepsilon(\bar{p})$ instead of ∂_1

and ∂_2. Note that $\partial_1\backslash U_\varepsilon(\bar{p})$ and $\partial_2\backslash U_\varepsilon(\bar{p})$ are compact and

$$d(\partial_1\backslash U_\varepsilon(\bar{p}), \partial_2) > 0, \; d(\partial_1, \partial_2\backslash U_\varepsilon(\bar{p})) > 0.$$

Suppose one or both of the above inequality does/do not hold. Without loss of generality let $d(\partial_1\backslash U_\varepsilon(\bar{p}), \partial_2) = 0$. We can find a convergent sequence $\{p_n\}_{n=1}^{\infty} \subseteq \partial_1\backslash U_\varepsilon(\bar{p})$ such that $d(p_n, \partial_2\backslash U_\varepsilon(\bar{p})) < 1/n$, where $d(p, \partial_2\backslash U_\varepsilon(\bar{p}))$ denotes the shortest distance from p to $\partial_2\backslash U_\varepsilon(\bar{p})$ and which is a continuous function of p. The limit of this sequence p^* satisfies $d(p^*, \partial_2) = 0$ and hence $p^* \in \partial_2\backslash U_\varepsilon(\bar{p})$. Thus we have $BR(p^*) = \{1, 2, 3\}$ but $p^* \neq \bar{p}$. Since there are only three pure strategies the set $BR^{-1}\{1, 2, 3\}$ must be a singleton. This results in a contradiction.

Now choose $T(\varepsilon)$ large enough so that

$$1/T(\varepsilon) < \min \{d(\partial_1\backslash U_\varepsilon(\bar{p}), \partial_2), d(\partial_1, \partial_2\backslash U_\varepsilon(\bar{p}))\}.$$

Using the same reasoning as in the proof of lemma 3 we can show that for $t > T(\varepsilon)$, if the three conditions in the present lemma are satisfied we can find a $t^* > t$ such that $B(p(t+k)) = B(p(t))$ for $0 \leq k \leq t^* - t - 1$ and $B(p(t^*)) = j$.

Q.E.D.

Proof of lemma 5: Since we know that whenever beliefs generated by fictitious play converge, they must converge to some Nash equilibrium, it suffices to show that p(t) and q(t) converge.

If there exists a $T_1 > T$ such that $B(p(t)) = i_1$ and $B(q(t)) = j_1$ for all $t > T_1$, then p(t) and q(t) inevitably converge to δ_{i_1} and δ_{j_1} respectively. So suppose there exists an infinite subsequence $\{p(t_n)\}_n$ such that $B(p(t_n)) = i_2$ for all t_n. By hypothesis of the lemma, $B(q(t_n)) = j_2$ for all t_n. Consider the interval $[\delta_{j_1}, \delta_{j_2}]$. Since player 2 uses only j_1 and j_2 for all $t > T$ we have

$$\lim_{t\to\infty} d(p(t), [\delta_{j_1}, \delta_{j_2}]) = 0.$$

We claim that there exist a point $p_0 \in [\delta_{j_1}, \delta_{j_2}]$ such that $\{i_1, i_2\} \in BR(p_0)$. Suppose not, then either $BR^{-1}(i_1) \cap [\delta_{j_1}, \delta_{j_2}] = \emptyset$ or $BR^{-1}(i_2) \cap [\delta_{j_1}, \delta_{j_2}] = \emptyset$, or both. Since player 1 uses i_1 and i_2 infinitely number of times, we can certainly find two convergent subsequences $\{p(t_m)\}_{m=1}^{\infty}$ and $\{p(t_r)\}_{r=1}^{\infty}$, where $B(p(t_m)) = i_1$ and $B(p(t_n)) = i_2$ for all m, n ≥ 1. Let $p(t_m) \to p_1$ and $p(t_n) \to p_2$. Both limits belong to $[\delta_{j_1}, \delta_{j_2}]$. By upper-hemi continuity of the best-response correspondence BR(p) we have $i_1 \in BR(p_1)$ and $i_2 \in BR(p_2)$. This results in a contradiction.

We now distinguish between the two following cases.

(i) $\partial(i_1,i_2) \subseteq [\delta_{j_1}, \delta_{j_2}]$.

(ii) $\partial(i_1,i_2) \cap [\delta_{j_1}, \delta_{j_2}] = \{p_0\}$;

Consider case (i) first. If $\partial(i_1,i_2) \subseteq [\delta_{j_1}, \delta_{j_2}]$ then either

$$BR^{-1}(i_1) \cap S_1^3\backslash[\delta_{j_1}, \delta_{j_2}] = \emptyset$$

or

$$BR^{-1}(i_2) \cap S_1^3\backslash[\delta_{j_1}, \delta_{j_2}] = \emptyset.$$

In either case we have $p(t) \in [\delta_{j_1}, \delta_{j_2}]$ for $t \geq 1$ (otherwise player 1 will never use i_1 or i_2). In particular, for $t > T$, we have $p(t) \in \partial(i_1, i_2)$. Now if $i_1 > i_2$ (in terms of label) then $B(p(t)) = i_1$ for all

$t > T$. Otherwise $B(p(t)) = i_2$ for all $t > T$. In neither situation would player 1 switch between i_1 and i_2 indefinitely. Consequently case (i) can be ruled out.

We now turn to case (ii). We will show that $p(t)$ converges to p_0. Let $\{t_k\}_k$, $t_k > T$, be the subsequence of times when player 1 switches his strategy such that

$$p(t_k - 1) = i_1,\ p(t_k) = i_2 \qquad k \text{ is odd}$$

and

$$p(t_k - 1) = i_2,\ p(t_k) = i_1 \qquad k \text{ is even.}$$

For any arbitrary $\varepsilon > 0$, let $U_\varepsilon(p_0)$ be an open ball containing p_0. If $\{t_k\}_k$ contains a subsequence $\{t_{k_m}\}_m$ such th. both $p(t_{k_m} - 1)$ and $p(t_{k_m})$ do not belong to $U_\varepsilon(p_0)$, then we can find a subsequence of beliefs $\{p(t_r)\}_r$ where both $p(t_r - 1)$ and $p(t_r)$ do not belong to $U_\varepsilon(p_0)$ and either (1) $p(t_r - 1) = i_1$, $p(t_r) = i_2$ for all t_r or (2) $p(t_r - 1) = i_2$, $p(t_r) = i_1$ for all t_r. Suppose $\{p(t_r)\}_r$ satisfies (1). Without loss of generality we can assume that $p(t_r)$ converges to some $p'' \in [\delta_1, \delta_2] \backslash U_\varepsilon(p_0)$. As $B(p(t_r)) = i_2$ for all t_r we have $i_2 \in BR(p'')$ but $i_1 \notin BR(p'')$. On the other hand the subsequence $\{p(t_r - 1)\}_r$ also converges to p'' since $\lim_{n\to\infty} \|p(t_r) - p(t_r - 1)\| \to 0$. But $B(p(t_r - 1)) = i_2$ for all t_r and it follows that $i_2 \in BR(p'')$, which is a contradiction. The case where $\{p(t_r)\}_r$ satisfies (2) is similarly ruled out.

If $p(t_k - 1) \in U_\varepsilon(p_0)$ and $p(t_k) \notin U_\varepsilon(p_0)$, or $p(t_k - 1) \notin U_\varepsilon(p_0)$ and $p(t_k) \in U_\varepsilon(p_0)$, for an infinite number of t_ks, then we can find subsequence $\{p(t_r)\}_r$ such that one of the following conditions is satisfied:

(1) $p(t_r-1) \in U_\varepsilon(p_0)$, $p(t_r) \notin U_\varepsilon(p_0)$, $B(p(t_r-1)) = i_1$ and

$B(p(t_r)) = i_2$ for all t_r;

(2) $p(t_r-1) \notin U_\varepsilon(p_0)$, $p(t_r) \in U_\varepsilon(p_0)$, $B(p(t_r-1)) = i_1$ and

$B(p(t_r)) = i_2$ for all t_r;

(3) $p(t_r-1) \in U_\varepsilon(p_0)$, $p(t_r) \notin U_\varepsilon(p_0)$, $B(p(t_r-1) = i_2$, and

$B(p(t_r)) = i_1$ for all t_r;

(4) $p(t_r-1) \notin U_\varepsilon(p_0)$, $p(t_r) \in U_\varepsilon(p_0)$, $B(p(t_r-1) = i_2$, and

$B(p(t_r) = i_1$ for all t_r.

Without loss of generality assumes $p(t_r)$ converges to p". If $\{p(t_r)\}$ satisfies (1), then $p'' \in [\delta_{j_1}, \delta_{j_2}] \cap U_\varepsilon(p_0)$ and $i_2 \in BR(p'')$ and hence $i_1 \notin BR(p'')$. But $p(t_r-1)$ also converges to p" and since $i_1 \in B(p(t_r-1))$ for all t_r, we have $i_2 \in BR(p'')$. This results in a contradiction. Cases (2), (3) and (4) yield similar contradiction.

If the above possibilities are ruled out, there must exist a $t_{\bar{k}}$ such that $p(t_k-1) \in U_\varepsilon(p_0)$ and $p(t_k) \in U_\varepsilon(p_0)$ for all $t_k \geq t_{\bar{k}}$. Consider any $t_k \geq t_{\bar{k}}$. If k is odd then $B(p(t_k)) = i_2$ and $B(q(t_k)) = j_2$. Now $B(p(t_k+S)) = i_2$ for $0 \leq S \leq t_{k+1}-t_k-1$ by definition of the subsequence t_k and hence $B(q(t_k+S)) = j_2$ for $0 \leq S \leq t_{k+1}-t_k-1$. Therefore $p(t_k+S) \in [p(t_k), p(t_{k+1}-1)]$ for $0 \leq S \leq t_{k+1}-t_k-1$. Since both $p(t_k)$ and $p(t_{k+1}-1)$ belong to $U_\varepsilon(p_0)$, which is a convex set, we have $p(t_k+S) \in U_\varepsilon(p_0)$ for $0 \leq S \leq t_{k+1}-t_k-1$. The argument is similar when k is even. Since t_k is arbitrary we have shown that $p(t) \in U_\varepsilon(p_0)$ forall $t > t_{\bar{k}}$. As ε is also arbitrary this shows that p(t) converges to p_0.

Similar arguments shows that q(t) converges to some $q_0 \in [\delta_{i_1}, \delta_{i_2}]$. This proves the lemma.

Q.E.D.

Proof of lemma 6: Consider $p(t'+1)$ and $q(t'+1)$. We have $p(t'+1) \in [p(t'), \delta_{B(q(t'))}]$ and $q(t'+1) \in [q(t'), \delta_{B(p(t'))}]$. By convexity of $BR^{-1}(i)$ we have $B(p(t')) \in BR(p)$ for any $p \in [p(t'), \delta_{B(q(t'))}]$ and $B(q(t')) \in BR(q)$ for any $q \in [q(t'), \delta_{B(p(t'))}]$. If $B(p(t')) = 1$ or 3 then it is obvious that $B(p) = B(p(t'))$ for all $p \in [p(t'), \delta_{B(q(t'))}]$. Similarly if $B(q(t')) = 1$ or 3 we have $B(q) = B(q(t'))$ for all $q \in [q(t'), \delta_{P(p(t'))}]$. Therefore for any of these combinations we have $B(p(t'+1)) = B(p(t'))$ and $B(q(t'+1)) = B(q(t'))$. Continuing in this way we can see that $B(p(t)) = B(p(t'))$ and $B(q(t)) = B(q(t'))$ for all $t > t'$ and hence $p(t) \rightarrow \delta_{B(q(t'))}$ and $q(t) \rightarrow \delta_{B(p(t'))}$.

Now suppose $B(p(t'))$ or $B(q(t'))$ is equal to 2 but not both. Without loss of generality let $B(p(t')) = 2$. Since $B(q(t')) = 1$ or 3 player 2 will not change his strategy as long as player 1 uses 2. If $B(p) = 2$ for every $p \in [p(t'), \delta_{B(q(t'))})$ then we have $B(p) = B(p(t'))$ for any $p \in [p(t'), \delta_{B(q(t'))}]$ and player 1 will not change his strategy before player 1. Consequently we have $B(p(t)) = B(p(t'))$ and $B(q(t)) = B(q(t'))$ for all $t > t'$ and $p(t)$, $q(t)$ will converge to a pure-strategy equilibrium.

If there exists a $p' \in [p(t'), \delta_{B(q(t'))})$ such that $B(p') = 3$ then it follows that $B(p) = 3$ for any $p \in [p', \delta_{B(q(t'))}]$. Let $t_1 > t'$ be the first time (since t') player 1 changes his strategy to 3. If $B(q(t')) = 3$ then we have $B(p(t_1)) = B(q(t_1)) = 3$. Since $3 \in BR(\delta_{B(q(t'))})$ we have $B(p(t_1)) = B(q(t_1)) \in A(p(t_1))$ and from above $p(t)$, $q(t)$ will converge to the equilibrium (δ_3, δ_3).

If $B(q(t')) = 1$ then we have $B(p(t_1)) = 3$ and $B(q(t_1)) = 1$. If furthermore $3 \in A(q(t_1)) = A(q(t'))$ then $p(t)$ and $q(t)$ will converge. On the other hand, if $3 \notin A(q(t_1))$ then either $3 \in BR(\delta_3)$ or $3 \notin BR(\delta_3)$. In the first case player 2 will switch to strategy 3 at some time t_2 (since $[q(t_1), \delta_3] \cap B^{-1}(2) = \emptyset$) and $(p(t), q(t))$ will converge to the pure-strategy equilibrium (δ_3, δ_3). In the second case $BR(\delta_3) = \{2\}$ and player 2 will switch his strategy to 2 at some time t_3 (since $[q(t_1), \delta_3] \cap B^{-1}(3) = \emptyset$) and we have to consider two further possibilities.

If $3 \in BR(\delta_2)$ then we have $B(p(t_3)) = 3 \in A(q(t_3))$ and $B(q(t_3)) = 2 \in A(p(t_3))$. From above $p(t)$, $q(t)$ will converge to a pure-strategy equilirbium. If $3 \notin BR(\delta_2)$ (and hence $BR(\delta_2) = \{ 1 \}$) then player 1 will switch his strategy to 1 at some time t_4. Now we have $B(p(t_4)) = 1 \in A(q(t_4))$ and $B(q(t_4)) = 2 \in A(p(t_4))$. Moreover $B(q) = 2$ for every $q \in [q(t_4), \delta_1]$. Therefore $p(t)$, $q(t)$ will converge to the pure- strategy equilibrium (δ_1, δ_2).

Q.E.D.

Proof of lemma 7: Since $\tau(p, \delta_{B(q)}) < \tau(q, \delta_{B(p)})$ we can find a $\mu > 0$ so that $\tau(p, \delta_{B(q)}) + \mu < \tau(q, \delta_{B(p)})$. Let $T(\mu)$ satisfy $1/T(\mu) < \mu$.

Suppose for some $t' > T(\mu)$ we have $p(t') = p$ and $q(t') = q$. Let t'' be the first time (after t') one of the players changes his stragegy. Since both players do not change their actual responses between t' and t'' we have, by definition of fictitious play,

$$\|p(t'') - p(t')\| = \Sigma_{t=t'+1}^{t''} \|p(t) - p(t-1)\|$$

$$= \Sigma_{t=t'+1}^{t''} \frac{1}{t} \|p(t-1) - \delta_{B(q)}\|$$

$$= \left(\sum_{t=t'+2}^{t''} \frac{t'}{t(t-1)} + \frac{1}{t'+1}\right)\|p(t-1) - \delta_{B(q)}\|.$$

and

$$\|q(t'') - q(t')\| = \left(\sum_{t=t'+2}^{t''} \frac{t'}{t(t-1)} + \frac{1}{t'+1}\right)\|q(t-1) - \delta_{B(p)}\|.$$

Now suppose $B(p(t'')) = B(p(t')) = B(p)$ and $B(q(t'')) = B(\delta_{B(p)})$. This implies

$$\|p(t'') - p(t')\| \leq \phi(p(t'), \delta_{B(q(t'))}) = \phi(p, \delta_{B(q)}) \text{ and}$$

$$\|q(t'') - q(t')\| \geq \phi(q(t'), \delta_{B(p(t'))}) = \phi(q, \delta_{B(p)}).$$

Substituting the above expressions for $\|p(t'') - p(t')\|$ and $\|q(t'') - q(t')\|$ and combining the two inequalities we get

$$\tau(q, \delta_{B(p)}) \leq \sum_{t=t'+2}^{t''} \frac{t'}{t(t-1)} + \frac{1}{t'+1} \leq \tau(p, \delta_{B(q)}),$$

which is a contradiction.

On the other hand, if $B(p(t'')) = B(\delta_{B(q)})$ and $B(q(t'')) = B(\delta_{B(p)})$ then we have

$$\|p(t''-1) - p(t')\| \leq \phi(p, \delta_{B(q)}),$$

$$\|q(t''-1) - q(t')\| \leq \phi(q, \delta_{B(p)}) \text{ and}$$

$$\|p(t'') - p(t')\| \geq \phi(p, \delta_{B(q)}),$$

$$\|q(t'') - q(t')\| \geq \phi(q, \delta_{B(p)}).$$

The first two inequalities yield

$$\sum_{t=t'+2}^{t''-1} \frac{t'}{t(t-1)} + \frac{1}{t'+1} \leq \tau(p, \delta_{B(q)}) + \mu < \tau(q, \delta_{B(p)})$$

and the last two inequalities yield

$$\tau(p, \delta_{B(q)}) + \mu < \tau(q, \delta_{B(p)}) \leq \sum_{t=t'+2}^{t''} \frac{t'}{t(t-1)} + \frac{1}{t'+1}.$$

Together they imply $t'/(t''(t''-1)) > \mu$. But this contradicts the fact that $t'/(t''(t''-1)) < 1/T(\mu) < \mu$. Therefore we must have $B(p(t'')) = B(\delta_{B(q)})$ and $B(q(t'')) = B(q(t')) = B(q)$.

Q.E D.

Proof of lemma 9: Since (p_0, q_0) does not constitute a Nash equilibrium we have $\{1, 2, 3\}\backslash BR(p_0) \neq \emptyset$ or $\{1, 2, 3\}\backslash BR(q_0) \neq \emptyset$ or both. Without loss of generality let $\{1, 2, 3\}\backslash BR(p_0) \neq \emptyset$ and let q_0 assigns strictly positive probability to at least one strategy not in $BR(p_0)$.

Because $BR(p_0)$ contains at most two strategies, there exists an $\varepsilon_0 > 0$ such that $B(p) \in BR(p_0)$ for all $p \in U_{\varepsilon_0}(p_0)$. Let $q[i_3]$ be the probability assigned to pure strategy i_3 by belief q. Let $i_3 \in \{1, 2, 3\}\backslash BR(p_0)$ and suppose $q_0[i_3] > 0$. For any $\xi > 0$ define

$$\bar{\alpha}(\xi) = \sup \{\alpha \mid \alpha = q[i_3] \text{ for some } q \in U_\xi(q_0)\},$$

$$\underline{\alpha}(\xi) = \inf \{\alpha \mid \alpha = q[i_3] \text{ for some } q \in U_\xi(q_0)\},$$

$$\Delta = \{\xi > 0 \mid \bar{\alpha}(\xi) > \underline{\alpha}(\xi) > 0\}.$$

Consider any $\xi \in \Delta$. Suppose $q(t) \in U_\xi(q_0)$ and $p(t) \in U_{\varepsilon_0}(p_0)$. For any given $N > 0$, $p(t+S) \in U_{\varepsilon_0}(p_0)$ for $0 \leq S \leq N-1$ implies $q(t+S+1)[i_3] < q(t+S)[i_3]$ for $0 \leq S \leq N-1$, where $q(t+S)[i_3]$ is the weight attributed to i_3 by belief $q(t+S)$. Indeed,

$$q(t+S+1)[i_3] = \frac{t+S}{t+S+1} q(t+S)[i_3]$$

and hence

$$q(t+N)[i_3] = \frac{t}{t+N} q(t')[i_3]$$

If $q(t+N) \in U_\xi(q_0)$ then $q(t+N)[i_3] \geq \underline{\alpha}(\xi)$. Therefore, starting at t, provided player 1 never play i_1 while player 2's belief is still inside $U_\xi(q_0)$, the maximum number of periods the latter will stay within $U_\xi(q_0)$ is given by

$$N_m(t,\xi) = 1 + \left[\frac{(1 - \rho(\xi))t}{\rho(\xi)}\right],$$

where $\rho(\xi) = \underline{\alpha}(\xi)/\overline{\alpha}(\xi)$ and $[x]$ denotes the integer value of x. Note that $N_m(t, \xi)$ goes to 1 as ξ approaches 0.

Consider any ε and δ greater than 0. Let $\varepsilon_1 < \min\{\varepsilon, \varepsilon_0\}$. Then choose $\delta_1 \in \Delta$, $\delta_1 < \delta$ and $T(\varepsilon,\varepsilon_1,\delta,\delta_1)$ so that for any $t > T$ we have:

$$(\ln(t+N_m(t, \delta_1)+1) - \ln(t+2) + \frac{1}{t+1})(\varepsilon + M) < \min\{\varepsilon, \varepsilon_0\} - \varepsilon_1$$

where $M = \max_j \|p_0 - \delta_j\|$.

Now suppose $p(t') \in U_\varepsilon(p_0)$ and $q(t') \in U_\delta(q_0)$ for some $t' > T(\varepsilon,\delta)$. Let $N(t') - 1$ be the total number of consecutive periods (starting at t') during which player 1's belief stays within $U_\varepsilon(p_0)$. By lemma 7, the total distance moved by player 1's belief between t' and t'+N(t') is equal to or less than

$$(\ln(t'+N(t')+1) - \ln(t'+2) + \frac{1}{t'+1})(\varepsilon + M).$$

If $N(t') \leq N_m(t',\delta_1)$ then by virtue of our choice of δ_1 and $T(\varepsilon, \delta)$ the total distance moved will be less than $\min\{\varepsilon, \varepsilon_0\} - \varepsilon_1$, the shortest distance between $U_{\varepsilon_1}(p_0)$ and a point in the boundary of $U_\varepsilon(p_0)$. This implies that $p(t'+N(t'))$ must still lie in $U_\varepsilon(p_0)$, a contradiction. Consequently we must have $N(t') > N_m(t', \delta_1)$. But then we have $q(t'+N(t')) \notin U_\xi(q_0)$ and player 2's belief must get out of $U_\xi(q_0)$ first. This establishes our lemma.

Q.E.D.

Proof of Theorem 1. Suppose the game satisfies condition (A). Without loss of generality let the asymmetric pure-strategy Nash equilibrium be (2, 3) (or (δ_2, δ_3)). We distinguish between two cases.

(A1) $1 \in BR(1)$;

(A2) $1 \notin BR(1)$.

Consider the first case. We have a symmetric pure strategy Nash equilibrium (1, 1). If there is $t' > 0$ such that $B(p(t')) = 1$ and $B(q(t')) = 1$, or $B(p(t')) = 2$ and $B(q(t')) = 3$, or $B(p(t')) = 3$ and $B(q(t')) = 2$, then we have $B(p(t')) \in A(q(t'))$ and $B(q(t')) \in A(p(t'))$. By lemma 6 $(p(t), q(t))$ must converge to one of the pure strategy Nash equilibria.

If no such t' exist, then for all $t > 0$ we have either $B(p(t)) = 2$ and $B(q(t)) = 2$, or $B(p(t)) = 3$ and $B(q(t)) = 3$, or $B(p(t)) = 1$ and $B(q(t)) = 2$, or $B(p(t)) = 1$ and $B(q(t)) = 3$, or $B(p(t)) = 2$ and $B(q(t)) = 1$, or $B(p(t)) = 3$ and $B(q(t)) = 1$. Consider the following two subcases:

(A1.1) There does not exist a $\bar{q} \in S_1^3$ such that $BR(\bar{q}) = \{1, 2, 3\}$;

(A1.2) There exists a $\bar{q} \in S_1^3$ such that $BR(\bar{q}) = \{1, 2, 3\}$.

In the first case, we have $\partial(1,2) \cap \partial(1,3) \cap \partial(2,3) = \varnothing$ and hence one of the boundaries must be an empty set. If $\partial(1,2)$ or $\partial(1,3) = \varnothing$ then for any $p \in B^{-1}(2)$ or $B^{-1}(3)$ we have $[p, \delta_2] \cap B^{-1}(1) = [p, \delta_3] = \varnothing$.

If there is a t' such that $B(p(t')) = B(q(t')) = 2$ (or 3), then both players' beliefs will be moving toward δ_2 (δ_3). Let $t_1 > t'$ be the first time when one or both players first change their strategies.Since

$[p(t'), \delta_2] \cap B^{-1}(1) = [q(t'), \delta_2] \cap B^{-1}(1) = \varnothing$ the players can only change their strategies to 3, if they ever do so (if they never change their strategies this implies that $[p(t'), \delta_2) \cap B^{-1}(3) = [q(t'), \delta_2) \cap B^{-1}(3) = \varnothing$ and (2, 2) must be a pure-strategy equilibrium)Note that we cannot have $B(p(t_1)) = B(\delta_2) = 3$ and $B(q(t_1)) = 2$ (or $B(p(t_1)) = 2$ and $B(q(t_1)) = 3$) since this possibility has already been ruled out. Therefore we must have $B(p(t_1)) = B(q(t_1)) = 3$.

Now both players' beliefs will start moving toward δ_3. If $BR(\delta_3) = \{2, 3\}$ then we have $B(p(t_1)) = B(q(t_1)) \in A(p(t_1)) = A(q(t_1))$ and by lemma 6 $(p(t), q(t))$ will converge to the pure-strategy Nash equilibrium (δ_3, δ_3).

If $3 \in BR(\delta_3)$ then let $t_2 > t_1$ be the next time when one or both players change their strategies. The fact that $[p(t_1), \delta_3] \cap B^{-1}(1) = [q(t_1), \delta_3] \cap B^{-1}(1) = \varnothing$ means that the players will not change their strategy to 1. Also we cannot have $B(p(t_2)) = 2$ and $B(q(t_2)) = 3$ (or $B(p(t_2)) = 3$ and $B(q(t_2)) = 2$). Therefore we must have $B(p(t_2)) = B(q(t_2)) = 2$. Continuing in this manner, we can see that both players will only use pure strategies 2 and 3 and that they always change their strategies at the same time. By lemma 5, $(p(t), q(t))$ must converge to a Nash equilibrium.

Now suppose the situations $B(p(t)) = B(q(t)) = 2$ and $B(q(t)) = B(q(t)) = 3$ do not arise for all $t > 0$. Then we have for all $t > 0$, either $B(p(t)) = 1$ and $B(q(t)) = 2$, or $B(p(t)) = 1$ and $B(q(t)) = 3$, or $B(p(t)) = 2$ and $B(q(t)) = 1$, or $B(p(t)) = 3$ and $B(q(t)) = 1$. Since all of these configurations are similar we only need to consider one of them.

First note that for any $p \in B^{-1}(1)$, we have either $[p, \delta_2) \cap B^{-1}(2) \neq \emptyset$ or $[p, \delta_3) \cap B^{-1}(3) \neq \emptyset$. Suppose $[p, \delta_2] \cap B^{-1}(2) \neq \emptyset$ (this also implies that we have $\partial(1,3) = \emptyset$ and therefore $[q, \delta_1] \cap B^{-1}(3) = \emptyset$ for any $q \in B^{-1}(2)$).If $p(t) \in B^{-1}(1)$ and $q(t) \in B^{-1}(2)$ then player 1's belief will be moving towards δ_2 and that the interval $[p(t), \delta_2]$ will be crossing the region $B^{-1}(2)$. Player 2's belief will be moving towards δ_1 and $[q(t), \delta_1] \cap B^{-1}(3) = \emptyset$.

By lemma 3 we can find a T such that starting at $t > T$ if player 2 does not change his strategy before player 1 then the latter will switch to 2 at some time $t' > t$. So assume we already start at some $t > T$ and let $t_1 > t$ be the first time one or both players change their strategies. Notice that player 2 cannot change his strategy first because then we would have $B(p(t_1)) = 1$ and $B(q(t_1)) = 1$, a situation we have already ruled out. Similarly, player 1 cannot be the first to switch strategy or otherwise we have $B(p(t_1)) = B(q(t_1)) = 2$.

Therefore both players must change their strategies simultaneously at time t_1 and we have $B(p(t_1)) = 2$ and $B(q(t_1)) = 1$. By repeating the same argument we can show that the two players always change their strategies at the same time and that only two strategies, 1 and 2 , are used for all remaining periods. Therefore by lemma 5 $(p(t), q(t))$ must converge to an equilibrium.

If $[p, \delta_3) \cap B^{-1}(3) \neq \emptyset$ for any $p \in B^{-1}(1)$ the situation is a little different. Suppose we already start at some $t' > T$, where T is the number defined in the same way as in lemma 3, and that $B(p(t')) = 1$ and $B(q(t')) = 2$. If $BR(\delta_2) = \{1, 3\}$ (which also implies $\partial(1,2) = \emptyset$), then $[p(t'), \delta_2) \cap B^{-1}(3) = \emptyset$ and player 1 will never change his strategy simultaneously with player 2. Let t_1 be the first time player 2 changes his strategy. We must have $B(p(t_1)) = 1$ and $B(q(t_2)) = 3$. player 1's belief will start moving towards δ_3 while player 1's belief will continue to move towards δ_1. Let t_2 be the next time one or both players change their strategies. By ruling out the impossible configurations we get $B(p(t_2)) = 3$ and $B(q(t_2)) = 1$, which is exactly the reverse of the situation at time t_1. By repeating the argument one can actually show that both players will always switch their strategies between 1 and 3 simultaneously.Again by lemma 5 $p(t)$ and $q(t)$ will converge. The case where $1 \notin BR(\delta_2)$ yields the same limiting outcomes.

Now consider the case where $\partial(2,3) = \emptyset$. We have for any $p \in B^{-1}(2)$and $q \in B^{-1}(3)$, $[p, \delta_2] \cap B^{-1}(1) \neq \emptyset$ and $[q, \delta_3] \cap B^{1}(1) \neq \emptyset$. Again by lemma 3 we can find a T such that starting at any $t > T$ if $[p(t), \delta_3] \cap B^{-1}(1) \neq \emptyset$ and that player 2 will not change his strategy before player 1, then player 1 will switch to 1 at some time t'. Suppose we start at some $t' > T$. If $B(p(t')) = B(q(t')) = 2$ (or 3) then both players' belief will be moving toward δ_2 and we have $[p(t'), \delta_2] \cap B^{-1}(1) \neq \emptyset$, $[q(t'), \delta_2] \cap B^{-1}(1) \neq \emptyset$. Let t_1 be the first time one or both players change their strategies. Note that we cannot have both players changing their strategies at the same time since this will imply that $B(p(t_1)) = B(q(t_1)) = 1$, a situation we have ruled out.

So only one of the players will change his strategy first. Without loss of generality assume player 1 change his stragtegy first.Then we have $B(p(t_1)) = 1$ and $B(q(t_1)) = 2$. Now player 1's belief will continue moving towards δ_2 but player 2's belief will be movingtowards δ_1. Also $[p(t_1), \delta_2] \cap B^{-1}(2) = \emptyset$ and $[q(t_1), \delta_1] \cap B^{-1}(1) = \emptyset$.Let t_2 be the next time one or both players change their strategies. We cannot have player 1 or 2 change his strategy first since this implies $B(p(t_2)) = 3$ and $B(q(t_2)) = 2$ or $B(p(t_2)) = B(q(t_2)) = 1$. Consequently both players must change their strategies simultaneouslyand we have $B(p(t_2)) = 3$ and $B(q(t_2)) = 1$.

Now $[p(t_2), \delta_1] \cap B^{-1}(2) = \varnothing$ and $[q(t_2), \delta_1] \cap B^{-1}(3) = \varnothing$. Let t_3 be the next time one or both players change their strategies. For the same reasons as above we cannot have one of the players change his strategy first. Therefore both players change simultaneously and $B(p(t_3)) = 1$ and $B(q(t_3)) = 2$. We have returned to the same situation as we have at time t_1. Repeating in this manner we can show that both players always change thier strategies at the same time and that player 1 will only use 1 and 3 and player 2 will only use 1 and 2 for the remaining periods. By lemma 5 $(p(t), q(t))$ will converge to an equilibrium. This discussion also covers the case where $B(p(t')) = 1$ and $B(q(t')) = 2$ (or $B(p(t')) = 3$ and $B(q(t')) = 1$) for some $t' > T$. All other cases are similar.

We now turn to (A1.2) where there exists a $\bar{p}$ such that $BR(\bar{p}) = \{1, 2, 3\}$. We further distinguish between two possibilities:

(1) $\bar{p} = \delta_i$ for some i, and (2) $\bar{p} \neq \delta_i$ for $i = 1, 2, 3$.

Consider (1) first. If $\bar{p} = \delta_1$ and $B^{-1}(1) = \varnothing$ both players will never use pure strategy 1. Therefore we need only consider the cases where $B(p(t')) = B(q(t')) = 2$ and $B(p(t')) = B(q(t')) = 3$ at some time t'. If in addition $B^{-1}(2) = \varnothing$ then both players will only use strategy 3 and $p(t)$, $q(t)$ will converge to the pure-strategy equilibrium (δ_3, δ_3). If $B^{-1}(2) \neq \varnothing$ but $BR(\delta_2) = \{2, 3\}$ then $p(t)$ and $q(t)$ will converge to the pure-strategy equilibrium (δ_2, δ_3). If $B^{-1}(2) \neq \emptyset$ and $BR(\delta_2) = \{3\}$ or $\{1, 3\}$ then it is easy to show that in both cases the two players will always switch their strategies together between 2 and 3 and hence $(p(t), q(t))$ must converge to a mixed-strategy equilibrium by lemma 5.

If $\bar{p} = \delta_1$ but $B^{-1}(1) \neq \emptyset$ then there must exist $p_1, p_2 \in [\delta_2, \delta_3]$, $p_1 \neq p_2$, such that for any $p \in [p_1, p_2]$ we have $B(p) = 1$. This implies that for any $p \in B^{-1}(1)$ and $q \in B^{-1}(2)$ (or $B^{-1}(3)$) we have $[p, \delta_2] \cap B^{-1}(2) = [p, \delta_3] \cap B^{-1}(3) = \emptyset$ and $[q, \delta_1] \cap B^{-1}(1) = \emptyset$. If $BR(\delta_3) = \{2\}$ but $BR(\delta_2) = \{1, 3\}$ then (δ_1, δ_2) is a pure-strategy equilibrium and by lemma 6 $B(p(t')) = 1$ and $B(q(t')) = 2$ (or $B(p(t')) = 2$ and $B(q(t')) = 1$) at some time t' implies $(p(t), q(t))$will converge to the pure-strategy equilibrium (δ_1, δ_2). When we have $B(p(t')) = 1$ and $B(q(t')) = 3$ (or $B(p(t')) = 3$ and $B(q(t')) = 1$) then it can be shown that $B(p(t'')) = 2$ and $B(q(t'')) = 3$ (or $B(p(t'')) = 3$ and $B(q(t'')) = 2$) for some $t'' > t'$ (because $[p(t'), \delta_3] \cap B^{-1}(3) = \emptyset$ and $[q(t'), \delta_1] \cap B^{-1}(2) = [q(t'), \delta_1] \cap B^{-1}(1) = \emptyset$) and hence $(p(t), q(t))$ will converge to the equilibrium (δ_2, δ_3).

If $BR(\delta_2) = \{3\}$ but $BR(\delta_3) = \{1, 2\}$ then $B(p(t')) = 1$ (2) and $B(q(t')) = 2$ (1) at some t' implies that beliefs will converge to the equilibrium (δ_2, δ_3). When $B(p(t')) = 1$ (3) and $B(q(t')) = 3$ (1) at some time t' beliefs will converge to the equilibrium (δ_1, δ_3).

If $BR(\delta_2) = \{1, 3\}$ and $BR(\delta_3) = \{1, 2\}$ then $B(p(t')) = 1$ (2) and $B(q(t')) = 2$ (1) at some t' implies that beliefs will converge to the equilibrium (δ_1, δ_2). When $B(p(t')) = 1$ (3) and $B(q(t')) = 3$ (1) at some t' beliefs will converge to the equilibrium (δ_1, δ_2).

Finally, if $BR(\delta_2) = \{3\}$ and $BR(\delta_1) = \{2\}$ then $B(p(t')) = 1$ (2) and $B(q(t')) = 2$ (1), or $B(p(t')) = 1$ (3) and $B(q(t')) = 3$ (1) at some t' implies that we have $B(p(t'')) = 2$ (3) and $B(q(t'')) = 3$ (2) at some $t'' > t'$.

Thus we can see that whenever one player's belief falls into the region $B^{-1}(1)$ while the other's falls into $B^{-1}(2)$ or $B^{-1}(3)$ we have either $p(t)$ and $q(t)$ converge to an equilibrium or beliefs will evolve to some configuration we have ruled out at the beginning of the proof. Now suppose for all t either $B(p(t)) = B(q(t)) = 2$ or $B(p(t)) = B(q(t)) = 3$ holds. Since neither player will ever use pure strategy 1 we have $\lim_{t\to\infty} p(t)[1] = \lim_{t\to\infty} q(t)[1] = 0$, where $p(t)[1]$ and $q(t)[1]$ are the probabilities attributed to pure strategy 1 by $p(t)$ and $q(t)$ respectively. Fix any small $\varepsilon > 0$ so that $p[1] > 1 - \varepsilon > 0$ for any $p \in U_\varepsilon(\bar{p})$. By lemma 4 we can find a $T(\varepsilon)$ such that for all $t > T$, if $[p(t), \delta_{B(q(t))}] \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) \neq \emptyset$ (or $[q(t), \delta_{B(p(t))}] \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) \neq \emptyset$) and that player 2 (player 1) never change his strategy before player 1 (player 2) then the latter will use strategy 1 at some time t''.

Let T' be such that $p(t)[1] < 1 - \varepsilon$ and $q(t)[1] < 1 - \varepsilon$ for all $t > T$. Consider any $t' > \max\{T', T(\varepsilon)\}$. Suppose $B(p(t')) = B(q(t')) = 2$ (the case $B(p(t')) = B(q(t')) = 3$ is similar). We have $[p(t'), \delta_2] \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) \neq \emptyset$ and that player 2 will not change his strategy before player 1 does. It follows that there is a t_1 such that $B(p(t_1)) = 1$. This contradicts the fact that player 1 will never use strategy 1.

Next consider the case where $\bar{p} = \delta_2$. Instead of considering beliefs at some arbitrary time it would be more convenient to work with beliefs at period 1. As before we would skip the cases $B(p(1)) = B(q(1)) = 1$,

$B(p(1)) = 2$ and $B(q(1)) = 3$, $B(p(1)) = 3$ and $B(q(1)) = 2$, since by lemma 6 beliefs inevitably converge in all these cases.

If $BR(\delta_3) = \{2, 3\}$ (hence $B(\delta_3) = 3$) then it is obvious that $B(p(1)) = B(q(1)) = 2$ or 3 implies that $p(t)$ and $q(t)$ will converge to the equilibrium (δ_3, δ_3). If $B(p(1)) = 1$ and $B(q(1)) = 3$ or $B(p(1)) = 3$ and $B(q(1)) = 1$ then either we have $B(p(t^*)) = B(q(t^*)) = 1$ or 3 at some time $t^* > t$, or if this doesn't occur then the two players will change their strategies simultaneously between 1 and 3 for an infinite number of times. By lemma 5 $(p(t), q(t))$ will converge to some mixed-strategy equilibrium.

If $B(\delta_3) = 2$ but $B^{-1}(3) = \{\delta_2\}$ then it is easy to show that either we have $(p(t), q(t))$ converging to one of the pure-strategy equilibria (δ_1, δ_1), (δ_2, δ_2) or (δ_2, δ_3), or the two players switch between 1 and 2 simultaneously for an infinite number of times, in which case we have $(p(t), q(t))$ converging to a mixed-strategy equilibrium.

If $B(\delta_3) = 2$ and $B^{-1}(3) \neq \{\delta_2\}$ the limiting outcomes are similar to those described above except for one thing. When $p(t)$ and $q(t)$ do not converge to a pure-strategy equilibrium the two players will switch between 1 and 3 infinitely often and in a simultaneous way and their beliefs will converge to a different mixed-strategy equilibrium. Finally, the case where $\bar{p} = \delta_3$ is exactly the same and we omit the details.

We now turn to (2) where $\bar{p}$ is not one of the vertices of S_1^3. It would be much simpler to start with beliefs at period 1. If $B(p(1)) = B(q(1)) = 1$, or $B(p(1)) = 2$ and $B(q(1)) = 3$, or $B(p(1)) = 3$ and $B(q(1)) = 2$ then by lemma 6 $p(t)$, $q(t)$ will converge to one of the pure-strategy equilibria. For other combinations of actual responses we need to distinguish between the following cases.

First, if $B^{-1}(1) = \varnothing$ then no player will ever use strategy 1. If $BR(\delta_2) = \{2, 3\}$ or $BR(\delta_3) = \{2, 3\}$ then it follows from lemma 6 that $B(p(1)) = B(q(1)) = 2$ or 3 implies $p(t)$, $q(t)$ will converge to one of the pure-strategy equilibria. If $BR(\delta_2) = \{3\}$ and $BR(\delta_3) = \{2\}$ then it is easy to see that either we have $B(p(t^*)) = 2$ and $B(q(t^*)) = 3$ or $B(p(t^*)) = 3$ and $B(p(t^*)) = 2$ at some t^* (and hence $(p(t), q(t))$ will converge to the asymmetric equilibrium (δ_2, δ_3)), or the two players always use identical strategies and they alternate between strategies 2 and 3 infinitely many times, in which case $(p(t), q(t))$ will converge to a mixed-strategy equilibrium.

Second, if $B^{-1}(1) \neq \varnothing$ but $B^{-1}(2) = \varnothing$ then players will never use pure strategy 2. Note that in this case we have $BR(\delta_3) = \{2, 3\}$ and there are only three pure-strategy equilibria, (δ_1, δ_1), (δ_2, δ_3) and (δ_3, δ_3). It is obvious that either we have $B(p(t^*)) = B(q(t^*)) = 1$ or $B(p(t^*)) = B(q(t^*)) = 3$ at some time t^*, or the two players never use the same strategy and they alternate between 1 and 3 infinitely many times. In both cases $p(t)$ and $q(t)$ will certainly converge.

Third, suppose both $B^{-1}(1)$ and $B^{-1}(2)$ are nonempty. If $BR(\delta_3) = \{2, 3\}$ then $B(p(1)) = B(q(1)) = 2$ or 3 implies that $(p(t), q(t))$ will converge to (δ_3, δ_3). When $B(p(1)) = 1$ and $B(q(1)) = 2$ (or $B(p(1)) = 2$ and $B(q(1)) = 1$) We have either $B(p(t^*)) = B(q(t^*)) = 1$ or $B(p(t^*)) = 3$ and $B(q(t^*)) = 2$ at some time t^*, or player 1 will alternate between strategies 1 and 3 and player 2 between 1 and 3 infinitely often and in a simultaneous way. In the later case $(p(t), q(t))$ will converge to a mixed-strategy equilibrium by lemma 5. The outcomes are similar when $B(p(1)) = 1$ and $B(q(1)) = 3$ (or $B(p(1)) = 3$ and $B(q(1)) = 1$).

If $BR(\delta_2) = \{2, 3\}$ and $\partial(2,3) \cap (\delta_2, \delta_3] \neq \emptyset$ then we have $\bar{p} \in (\delta_2, \delta_3)$. In this case $B(p(1)) = B(q(1)) = 2$ or 3 implies that both players will always use identical strategies and they alternate between strategies 2 and 3 for infinitely many times. If one player uses strategy 1 while the other uses strategy 2 or 3 at period 1 then it is not difficult to verify that at some time $t' > 1$ we will have $B(p(1)) = B(q(1)) = 1$. Therefore $(p(t), q(t))$ converge for all initial beliefs.

If $BR(\delta_2) = \{2, 3\}$ but $\partial(2,3) \cap (\delta_2, \delta_3] = \emptyset$ then $B(p(1)) = B(q(1)) = 2$ implies that $(p(t), q(t))$ will converge to (δ_2, δ_2). If in addition $B(\delta_1) = \{1\}$ then $B(p(1)) = B(q(1)) = 3$ implies that $p(1) = q(1) = \delta_2$ and we have $B(p(2)) = B(q(2)) = 2$. Thus beliefs will converge to (δ_2, δ_2). When $B(p(1)) = 1$ and $B(q(1)) = 2$ (or $B(p(1)) = 2$ and $B(p(1)) = 1$) we have either $p(t)$, $q(t)$ converging to a pure-strategy equilibrium $((\delta_1, \delta_1)$ or $(\delta_2, \delta_3))$ or the two players will be alternating between two pairs of strategies (player 1 between 1 and 3 and player 2 between 1 and 3) for an infinite number of times, in which case $(p(t), q(t))$ will converge to a mixed- strategy equilibrium. When $B(p(1)) = 1$ and $B(q(1)) = 3$ (or $B(p(1)) = 3$ and $B(q(1)) = 1$) the outcomes are similar.

If $BR(\delta_2) = \{2, 3\}$ and $\partial(2,3) \cap (\delta_2, \delta_3] = \emptyset$ but $BR(\delta_1) = \{1, 3\}$ then $B(p(1)) = B(q(1)) = 3$ may imply either $p(1) = q(1) = \delta_2$, or $p(1) = q(1) = \delta_1$, or $p(1) = \delta_1$ and $q(1) = \delta_2$, or $p(1) = \delta_2$ and $q(1) = \delta_1$. In the first two cases beliefs will converge to (δ_1, δ_1) and (δ_2, δ_2). The limiting outcomes of the next two cases are however less obvious. Instead of tracing the evolution of beliefs starting from period 1 we would again consider the behavior of beliefs at arbitrary t.

Suppose $B(p(t')) = B(q(t')) = 2$ at some time t'. If $[p(t'), \delta_2) \cap B^{-1}(3) = [q(t'), \delta_2) \cap B^{-1}(3) = \varnothing$ then $(p(t), q(t))$ will converge to (δ_2, δ_2). If either of the intersections, or both, is not not empty then one or both of the players will switch to strategy 2 at some future time. Let t_1 be the time when this happens. If $B(p(t_1)) = B(q(t_1)) = 3$ then we have $B(p(t_1+1)) = B(q(t_1+1)) = 2$ and $[p(t_1+1), \delta_2) \cap B^{-1}(3) = [q(t_1+1), \delta_2) \cap B^{-1}(3) = \varnothing$. Therefore $(p(t), q(t))$ will converge to (δ_2, δ_2). Otherwise we have $B(p(t_1)) = 2$ and $B(q(t_1)) = 3$ or $B(p(t_1)) = 3$ and $B(q(t_1)) = 2$ and beliefs will converge to (δ_2, δ_3).

Suppose $B(p(t')) = 1$ and $B(q(t')) = 3$ (or $B(p(t')) = 3$ and $B(q(t')) = 1$). If $[p(t'), \delta_3) \cap B^{-1}(2) = \varnothing$ then $1 \in BR(\delta_3)$ and we have $B(p(t')) \in A(q(t'))$ and $B(q(t')) \in A(p(t'))$. On the other hand, if $[p(t'), \delta_3) \cap B^{-1}(2) \neq \varnothing$ then we have $B(p(t'')) = 2$ and $B(q(t'')) = 3$ at some $t'' > t'$. By lemma 6 beliefs will converge in both cases.

Now suppose $B(p(t')) = 1$ and $B(q(t')) = 2$ (or $B(p(t')) = 2$ and $B(q(t')) = 1$) at some t'. Since $[p(t'),\delta_2) \cap B^{-1}(2) \neq \varnothing$ or $[p(t'), \delta_2) \cap B^{-1}(3) \neq \varnothing$, and $[q(t'), \delta_1) \cap B^{-1}(1) \neq \varnothing$ or $[q(t'), \delta_1) \cap B^{-1}(3) \neq \varnothing$ at least one of the players will change his strategy at some future time. We know that if there is a $t'' > t'$ such that $B(p(t'')) = B(q(t'')) = 1$, 2 or 3, or $B(p(t'')) = 2$ and $B(q(t'')) = 3$ or $B(p(t'')) = 3$ and $B(q(t'')) = 2$, or $B(p(t'')) = 1$ and $B(q(t'')) = 3$, or $B(p(t'')) = 3$ and $B(q(t'')) = 1$, then beliefs will converge to an equilibrium. If none of the above configurations of actual responses ever occurs, then it must be the case that the two players always alternate between strategies 1 and 2 simultaneously and they do so for an infinite number of times.

Again by lemma 5 $(p(t), q(t))$ must converge to a mixed-strategy equilibrium.

Finally, the analysis of the case where $BR(\delta_2) = \{3\}$ and $BR(\delta_3) = \{2\}$ is very similar and is omitted for brevity.

This completes our examination of games that belong to (A1), we now proceed to category (a2) where $1 \notin BR(\delta_1)$.

When $1 \notin BR(\delta_1)$ we have either $BR(\delta_1) = \{2\}$, $BR(\delta_1) = \{3\}$ or $BR(\delta_1) = \{2, 3\}$. All of these cases are qualitatively the same and it is sufficient to discuss only one of them, say, $BR(\delta_1) = \{2\}$. If $B^{-1}(1) = \varnothing$ then players will use only strategies 2 and 3 and, as we have shown before, beliefs will converge to some Nash equilibrium.

So suppose $B^{-1}(1) \neq \varnothing$. As in (a1) we would address the following two subcases separately:

(A2.1) There does not exist a $\bar{p}$ such that $BR(\bar{p}) = \{1, 2, 3\}$;

(A2.2) There exists a $\bar{p}$ such that $BR(\bar{p}) = \{1, 2, 3\}$.

Consider (A2.1) first. Since $B(\delta_1) = 2$ and $2 \in BR(\delta_3)$ we have $[p, \delta_1) \cap B^{-1}(1) \neq \varnothing$ and $[p, \delta_3) \cap B^{-1}(1) \neq \varnothing$ for every $p \in B^{-1}(3)$, and $[q, \delta_2) \cap B^{-1}(1) \neq \varnothing$ and $[q, \delta_2) \cap B^{-1}(1) \neq \varnothing$ for every $q \in B^{-1}(2)$. Now by lemma 3 we can find a T such that for all $t > T$, $[p(t), \delta_{B(q(t))}) \cap B^{-1}(1) \neq \varnothing$ and/or $[q(t), \delta_{B(p(t))}) \cap B^{-1}(1) \neq \varnothing$ implies $B(p(\bar{t})) = 1$ or $B(q(\bar{t})) = 1$, or both, for some $\bar{t} \geq t$. Consider any $t' > T$. If $B(p(t')) = 2$ and $B(q(t')) = 3$, or $B(p(t')) = 3$ and $B(q(t')) = 2$ then by lemma 6 $p(t), q(t)$ would converge to (δ_2, δ_3). If $B(p(t')) = 1$ and $B(q(t')) = 2$ (or $B(p(t')) = 2$ and $B(q(t')) = 1$) then because the sets $[p(t'), \delta_2) \cap B^{-1}(2)$, $[q(t'), \delta_1) \cap B^{-1}(1)$ and $[q(t'), \delta_1) \cap B^{-1}(3)$ are all empty we have $B(p(t'')) = 3$ and $B^{-1}(q(t'')) = 2$ for some $t'' > t'$ and hence $p(t)$,

$q(t)$ will converge.

If $B(p(t')) = B(q(t')) = 2$ then we have either $B(p(t'')) = 2$ and $B(q(t'')) = 1$, or $B(p(t'')) = 1$ and $B(q(t'')) = 2$, or $B(p(t'')) = B(q(t''))$ $= 1$, at some $t'' > t'$. In the first two cases $p(t)$, $q(t)$ will certainly converge. So suppose we have $B(p(t'')) = B(q(t'')) = 1$. Let t_1 be the next time when one or both players changes his strategy. We have either $B(p(t_1)) = 1$ and $B(q(t_1)) = 2$, or $B(p(t_1)) = 2$ and $B(q(t_1)) = 1$, or $B(p(t_1)) = B(q(t_1)) = 2$. Again the first two configurations will lead to convergence of beliefs. On the other hand if $B(p(t_1)) = B(q(t_1)) = 2$ we would return to the situation we have at t'. Continuing in this manner, we have either $(p(t), q(t))$ converges to the pure-strategy equilibrium (δ_2, δ_3) or the two players always use identical strategies and they alternate between pure strategies 1 and 2 infinitely many times, in which case beliefs will converge to a symmetric mixed-strategy equilibrium.

Now suppose $B(p(t')) = B(q(t')) = 3$. Let t_1 be the first time after t' when one of the players changes his strategy. If $B(p(t_1)) = B(q(t_1))$ $= 1$ then from above $(p(t), q(t))$ would converge. If $B(p(t_1)) = 3$ and $B(q(t_1)) = 1$ let t_2 be the next time one or both players changes his strategy. We have either $B(p(t_1)) = 3$ and $B(q(t_1)) = 2$, or $B(p(t_1)) =$ $B(q(t_1)) = 1$, or $B(p(t_1)) = 1$ and $B(q(t_1)) = 2$. From above each of these configurations leads to convergence of beliefs. If $B(p(t_1)) = 1$ and $B(q(t_1)) = 3$ the results are similar.

This shows that fictitious play always converges for subcase (A2.1). We now consider subcase (A2.2). First note that, as in subcase (A2.1), if $B(p(t')) = 2$ and $B(q(t')) = 3$, or $B(p(t')) = 3$ and $B(q(t'))$

= 2, or $B(p(t')) = 1$ and $B(q(t')) = 2$, or $B(p(t')) = 2$ and $B(q(t')) = 1$ for some $t' > 0$ then $(p(t), q(t))$ will converge to the pure-strategy equilibrium (δ_2, δ_2). So suppose none of these combinations of actual responses ever occur. This means that for any $t > 0$ we have either $B(p(t)) = B(q(t)) = 1, 2$ or 3, or $B(p(t)) = 1$ and $B(q(t)) = 3$, or $B(p(t)) = 3$ and $B(q(t)) = 1$.

It is useful to differentiate between the following two possibilities:

(1) There is a subsequence $\{p(t_m)\}$ or $\{q(t_m)\}$ such that $p(t_m) \longrightarrow \bar{p}$ or $q(t_m) \longrightarrow \bar{q}$;

(2) No such subsequence exists.

Consider possibility (2) first. In this case we can find an $\varepsilon > 0$ and a $\hat{T}$ such that $p(t), q(t) \in S_1 \backslash B_\varepsilon(\bar{p})$ (or $p(t), q(t) \in S_1 \backslash B_\varepsilon(\bar{q})$) for all $t > \hat{T}$. On the other hand by lemma 4 there exists a $T(\varepsilon)$ such that starting at any $t > T(\varepsilon)$, if $(p(t), \delta_{B(q(t))}) \cap B^{-1}(1) \backslash U_\varepsilon(\bar{p}) \neq \varnothing$ (or $(q(t), \delta_{B(p(t))}) \cap B^{-1}(1) \backslash U_\varepsilon(\bar{p})) \neq \varnothing$) and player 2 (player 1) does not change his strategy before player 1 (2), then player 1 (2) will switch to 1 at some time $\bar{t} > t$.

Let us start at some arbitrary $t' > \mathrm{Max}\ \{\hat{T}, T(\varepsilon)\}$ and let $\{t_k\}$, where $t_k > t'$, be the subsequence of times when one or both of the players changes his strategy. If there exists a $\hat{T}$ such that for all $t_k > \hat{T}$, $[p(t_k), \delta_{B(q(t_k))}) \cap B^{-1}(1) \backslash U_\varepsilon(\bar{p}) = \varnothing = [q(t_k), \delta_{B(p(t_k))}) \cap B^{-1}(1) \backslash U_\varepsilon(\bar{p})$ then we have $B(p(t)) = B(q(t)) = 2$ or 3 for all $t > \hat{T}$. It follows from lemma 5 that $(p(t), q(t))$ will converge to a mixed-strategy equilibri_m. If no such $\hat{T}$ exists then we have

$$[p(t_k),\ \delta_{B(q(t_k))}) \cap B^{-1}(1)\setminus U_\varepsilon(\bar{p}) \neq \emptyset \tag{1}$$

or

$$[q(t_k),\ \delta_{B(p(t_k))}) \cap B^{-1}(1)\setminus U_\varepsilon(\bar{p}) \neq \emptyset \tag{2}$$

or both for infinitely many t_ks. This implies that for some $t_{k'}$ we have either $B(p(t_{k'})) = 1$ or $B(q(t_{k'})) = 1$ or both. To analyse the limiting behavior of beliefs we need to distinguish between the following two situations:

(i) $\exists\ p_1, p_2 \in [\delta_1, \delta_2] \cap B^{-1}(1)$ where $p_1 \neq p_2$;

(ii) $\exists\ q_1, q_2 \in [\delta_1, \delta_2] \cap B^{-1}(1)$ where $q_1 \neq q_2$.

Consider (i) first. Suppose $B(p(t_{k'})) = B(q(t_{k'})) = 1$. By lemma 1 we have $[p(t_{k'+1}), \delta_1) \cap B^{-1}(3) = \emptyset = [q(t_{k'+1}), \delta_1) \cap B^{-1}(3)$ and hence $B(p(t_{k'+2})) = B(q(t_{k'+2})) = 2$. Now using simple geometry one can show that

$$[p(t_{k'+2}),\ \delta_2) \cap B^{-1}(1)\setminus U_\varepsilon(\bar{p}) \neq \emptyset$$

and

$$[q(t_{k'+2}),\ \delta_2) \cap B^{-1}(1)\setminus U_\varepsilon(\bar{p}) \neq \emptyset.$$

Consequently $B(p(t_{k'+3})) = B(q(t_{k'+3})) = 1$. Continuing in this manner we get

$$B(p(t_{k'+s})) = B(q(t_{k'+s})) = \begin{cases} 1 & s \text{ is odd} \\ 2 & s \text{ is even} \end{cases}.$$

Therefore by lemma 5 $p(t)$, $q(t)$ will converge to a mixed-strategy equilibrium. In fact, whenever $B(p(t_0)) = B(q(t_0)) = 1$ for some $t_0 \geq t'$ $p(t)$, $q(t)$ will converge to that mixed-strategy equilibrium.

Now suppose $B(p(t_{k'})) = 1$ and $B(q(t_{k'})) = 3$ (or $B(p(t_{k'})) = 3$ and $B(q(t_{k'})) = 1$). If $[q(t_{k'}), \delta_1) \cap B^{-1}(1)\setminus U_\varepsilon(\bar{p}) \neq \emptyset$ then we have

$B(p(t_{k^*+1})) = B(q(t_{k^*+1})) = 1$ This is because $B(p(t_{k^*+1})) = 2$ and $B(q(t_{k^*+1})) = 1$, as well as $B(p(t_{k^*+1})) = 2$ and $B(q(t_{k^*+1})) = 3$, are combinations we have ruled out. On the other hand, if $B(p(t_{k^*+1})) = B(q(t_{k^*+1})) = 3$ then one can show that $[p(t_k), \delta_{q(t_k)}) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) = \varnothing = [q(t_k), \delta_{B(p(t_k))}) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p})$ for all $t_k \geq t_{k^*+1}$, which contradicts the hypothesis that (1) or (2) holds for infinitely many t_ks. Consequently we have $B(p(t_{k^*+1})) = B(q(t_{k^*+1})) = 1$. It follows from the discussion of the previous pharagraph that $(p(t), q(t))$ will converge to a mixed-strategy equilibrium.

If $[q(t_{k^*}), \delta_3) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) = \varnothing$ then we have $B(p(t_{k^*+1})) = B(q(t_{k^*+1})) = 2$. Consider $[p(t_{k^*+1}), \delta_2)$ and $[q(t_{k^*+1}), \delta_2)$. If $[p(t_{k^*+1}), \delta_2) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) = \varnothing = [q(t_{k^*+1}), \delta_2) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p})$ then again it can be shown that $[p(t_k), \delta_{q(t_k)}) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) = \varnothing$ and $[q(t_k), \delta_{B(p(t_k))}) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) = \varnothing$ for all $t_k \geq t_{k^*+1}$, which is impossible. So either $[p(t_{k^*+1}), \delta_2) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) \neq \varnothing$ or $[q(t_{k^*+1}), \delta_2) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) \neq \varnothing$ or both.

Now if

$$[p(t_{k^*+1}),\delta_2) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) \neq \varnothing \tag{3}$$

and

$$[q(t_{k^*+1}),\delta_2) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) \neq \varnothing \tag{4}$$

then we have $B(p(t_{k^*+2})) = B(q(t_{k^*+2})) = 1$. As before $(p(t), q(t))$can be shown to converge to a mixed-strategy equilibrium. If only (3) holds then we have $B(p(t_{k^*+2})) = 1$ and $B(q(t_{k^*+2})) = 3$, which is the same situation as we have at t_{k^*}. If only (4) holds then we have $B(p(t_{k^*+2})) = 3$ and $B(q(t_{k^*+2})) = 1$, which is symmetrical to the situation we have at $t_{k'}$.

We claim that for some $t_{k''} > t_{k'}$, $B(p(t_{k''})) = B(q(t_{k''})) = 1$. Suppose the contrary. Then from the preceding analysis we have, for all t_k, either $B(p(t_k)) = 1$ and $B(q(t_k)) = 3$, or $B(p(t_k)) = 3$ and $B(q(t_k)) = 1$, or $B(p(t_k)) = B(q(t_k)) = 2$. It is easy to verify that each of the three possible combinations occurs infinitely many times. Let $\{t_{k_r}\} \subseteq \{t_k\}$ be a subsequence of $\{t_k\}$ such that $B(p(t_{k_r}-1)) = 1$, $B(q(t_{k_r}-1)) = 3$, $B(p(t_{k_r})) = B(q(t_{k_r})) = 2$, $B(p(t_{k_r+1})) = 3$ and $B(q(t_{k_r+1})) = 1$ for all t_{k_r}. Without loss of generality let $p(t_{k_r}) \to p_0$. Since $\|p(t_{k_r}) - p(t_{k_r}-1)\| \to 0$ we have $p(t_{k_r}-1) \to p_0$ and hence $BR(p_0) = \{1, 2\}$.

Now by construction of $\{t_{k_r}\}$ we have

$$[p(t_{k_r}), \delta_2) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) = \varnothing$$

for all t_{k_r}, which implies that $[p_0, \delta_2) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) = \varnothing$ (Otherwise we could find an $\varepsilon' > 0$ so that $[p, \delta_2) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) \neq \varnothing$ for all $p \in U_{\varepsilon'}(p_0)$, which in turn yields the contradiction that $[p(t_{k_r}), \delta_2) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) \neq \varnothing$ for t_{k_r} large enough.) It follows from simple geometrical consideration that $[p_0,\delta_2) \cap U_\varepsilon(\bar{p}) \neq \varnothing$. Let $p'' \in [p_0, \delta_2) \cap U_\varepsilon(\bar{p})$ and $\|p'' - p_0\| > a > 0$.

Let $U_\delta(p_0)$ be a neighbourhood of p_0 such that for every $p \in U_\delta(p_0)$ satisfying $[p, \delta_2) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) = \varnothing$, there are $b_1, b_2 \in [p,\delta_2) \cap U_\varepsilon(\bar{p})$ where $\|b_1 - b_2\| > a$. Furthermore, specify T^* so that $t > T^*$ implies $\|p(t) - p(t-1)\| < a$. Now consider any $t_{k_r} > T^*$ so that $p(t_{k_r}) \in U_\delta(p_0)$. By our construction $B(p(t_{k_r+1}-1)) = 2$ and $B(p(t_{k_r+1})) = 3$ and

$\|p(t_{k_r+1}) - p(t_{k_r+1}-1)\| < a$. On the other hand, there exist b_1^*, $b_2^* \in [p(t_{k_r+1}-1),\ p(t_{k_r+1})] \cap U_\varepsilon(\bar{p})$ such that $\|b_1^* - b_2^*\| > a$, which is a contradiction. This verifies our claim that $B(p(t_{k^{**}})) = B(q(t_{k^{**}})) = 1$ for some $t_{k^{**}} > t^*$ and completes our analysis of (i).

We now go to (ii) where there exists two distinct points q_1, $q_2 \in [\delta_1,\ \delta_2] \cap B^{-1}(1)$. Consider $p(t_{k^*})$ and $q(t_{k^*})$ again. First note that if $B(p(t_{k^{**}})) = 3$ and $B(q(t_{k^{**}})) = 1$ (or $B(p(t_{k^{**}})) = 1$ and $B(q(t_{k^{**}})) = 3$) for some $t_{k^{**}} \geq t_{k^*}$ and $[p(t_{k^*}),\ \delta_1) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) = \varnothing$ (or $[q(t_{k^*}),\ \delta_1) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) = \varnothing$) then $(p(t),\ q(t))$ will converge to a mixed-strategy equilibrium. To prove this result we can start at $t_{k^{**}}$ and trace the evolution of the pair of actual responses. Given our conditions, it is easy to see that we must have $B(p(t_{k^{**}+1})) = 2$ and $B(q(t_{k^{**}+1})) = 2$ (since $[q(t_{k^{**}+1}),\ \delta_3) \cap B^{-1}(3) = \varnothing$ other combinations are infeasible or have been eliminated from our agenda). By simple geometry one can verify that $[p(t_{k^{**}+1}),\ \delta_2) \cap B^{-1}(1) = \varnothing$ and $[q(t_{k^{**}+1}),\ \delta_2) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) \neq \varnothing$ and hence $B(p(t_{k^{**}+2})) = 3$ and $B(q(t_{k^{**}+2})) = 1$. Now simple geometrical consideration indicates that $[p(t_{k^{**}+2}),\ \delta_2) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) = \varnothing$ and hence $B(p(t_{k^{**}+3})) = B(q(t_{k^{**}+3})) = 2$. Continuing in this manner one can show that the two players always change their strategies simultaneously and they alternate between strategies 1, 2 and 2, 3 respectively. Therefore by lemma 5 $(p(t),\ q(t))$ will converge to a mixed-strategy equilibrium.

We now claim that for some $t_{k^{**}} \geq t_{k^*}$ we have $B(p(t_{k^{**}})) = 3$ and $B(q(t_{k^{**}})) = 1$ (or $B(p(t_{k^{**}})) = 1$ and $B(q(t_{k^{**}})) = 3$) and $[p(t_{k^*}),\ \delta_1) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) = \varnothing$ (or $[q(t_{k^*}),\ \delta_1) \cap B^{-1}(1)\backslash U_\varepsilon(\bar{p}) = \varnothing$). Suppose not. Then for all t_k we have either $B(p(t_k)) = B(q(t_k)) = 1$, 2 or 3, or

$B(p(t_k)) = 3$ and $B(q(t_k)) = 1$ but $[p(t_k), \delta_1) \cap B^{-1}(1)\setminus U_\varepsilon(\bar{p}) \neq \varnothing$, or $B(p(t_k)) = 1$ and $B(q(t_k)) = 3$ but $[q(t_k), \delta_1) \cap B^{-1}(1)\setminus U_\varepsilon(\bar{p}) = \varnothing$. Furthermore the first three pairs will occur infinitely many times.

Let $\{t_{k_r}\}$ be a subsequence of $\{t_k\}$ such that $B(p(t_{k_r}-1)) = 1$, $B(p(t_{k_r})) = 2$ and $B(p(t_{k_r+1})) = 3$ for all t_{k_r} (or $B(p(t_{k_r}-1)) = 1$, $B(p(t_{k_r})) = 3$ and $B(p(t_{k_r+1})) = 2$ for all t_{k_r}. Without loss of generality let $p(t_{k_r}) \to p_0$ (and hence $p(t_{k_r}-1) \to p_0$). We have $BR(p_0) = \{1, 2\}$ (or $BR(p_0) = \{1, 3\}$). By our construction of $\{t_{k_r}\}$ we have $[p(t_{k_r}), \delta_2) \cap B^{-1}(1)\setminus U_\varepsilon(\bar{p}) = \varnothing$ and therefore $[p_0, \delta_2) \cap B^{-1}(1)\setminus U_\varepsilon(\bar{p}) = \varnothing$ (otherwise we can find a neighbourhood $U_{\varepsilon'}(p_0)$ such that for every $p \in U_{\varepsilon'}(p_0)$, $[p, \delta_2) \cap B^{-1}(1)\setminus U_\varepsilon(\bar{p}) \neq \varnothing$). It follows from simple geometry that $[p_0, \delta_2) \cap U_\varepsilon(\bar{p}) \neq \varnothing$. Let $p'' \in [p_0, \delta_2) \cap U_\varepsilon(\bar{p})$ and $\|p'' - p_0\| > a > 0$.

Let $U_\delta(p_0)$ be a neighbourhood of p_0 such that for every $p \in U_\delta(p_0)$ satisfying $[p, \delta_2) \cap B^{-1}(1)\setminus U_\varepsilon(\bar{p}) = \varnothing$, there are $b_1, b_2 \in [p, \delta_2) \cap U_\varepsilon(\bar{p})$ where $\|b_1 - b_2\| > a$. Furthermore, specify T^* so that $t > T^*$ implies $\|p(t) - p(t-1)\| < a$. Now consider any $t_{k_r} > T^*$ so that $p(t_{k_r}) \in U_\delta(p_0)$. By our construction $B(p(t_{k_r+1}-1)) = 2$ and $B(p(t_{k_r+1})) = 3$ and $\|p(t_{k_r+1}) - p(t_{k_r+1}-1)\| < a$. On the other hand, there exist $b_1^*, b_2^* \in [p(t_{k_r+1}-1), p(t_{k_r+1})] \cap U_\varepsilon(\bar{p})$ such that $\|b_1^* - b_2^*\| > a$, which is a contradiction. This verifies our claim and completes our analysis of (ii).

Next consider possibility (1) where there is a subsequence $\{p(t_m)\}$ (or $\{q(t_m)\}$) such that $p(t_m) \to \bar{p}$ (or $q(t_m) \to \bar{p}$). As in the analysis of (1) we distinguish between the following two cases:

(i) $\exists\ p_1, p_2 \in [\delta_1, \delta_2] \cap B^{-1}(1)$ where $p_1 \neq p_2$;

(ii) $\exists\ q_1, q_2 \in [\delta_1, \delta_2] \cap B^{-1}(1)$ where $q_1 \neq q_2$.

Consider (1) first. We first show that for any $\bar{q} \neq \bar{p}$ the pair $(\bar{p}, \bar{q})$ cannot be a limit point of the sequence $(\{p(t)\}, \{q(t)\})$. Suppose not. Then there exists a $\bar{q} \neq \bar{p}$ such that $(\bar{p}, \bar{q})$ or $(\bar{q}, \bar{p})$ is a limit point of $(\{p(t)\}, \{q(t)\})$. Without loss of generality let $(\{p(t_n)\}, \{q(t_n)\})$ be a subsequence where $p(t_n) \rightarrow \bar{p}$ and $q(t_n) \rightarrow \bar{q}$.

Suppose $BR(\bar{q}) = \{1, 3\}$. We note that for any $p \in B^{-1}(1)$ and $p' \in B^{-1}(3)$ where $[q, \delta_1] \cap B^{-1}(1) = \emptyset$ the numbers $\tau(p, \delta_1)$ and $\tau(p', \delta_1)$ are defined. Since

$$\inf \left\{ \|\bar{p} - p\| \mid p \in [\bar{p}, \delta_1] \cap B^{-1}(2) \right\} = 0$$

and

$$\inf \left\{ \|\bar{q} - q\| \mid q \in [\bar{q}, \delta_1] \cap B^{-1}(2) \right\} > 0$$

there exist $\mu > 0$ such that $q \in U_\mu(\bar{q}) \cap B^{-1}(1)$ and $p \in U_\mu(\bar{p}) \cap B^{-1}(1)$, or $p \in U_\mu(\bar{p}) \cap B^{-1}(3)$ where $[p, \delta_1] \cap B^{-1}(1) = \emptyset$, implies

$$\tau(p, \delta_1) + \mu < \tau(q, \delta_1).$$

Hence by lemma 7 we can find a $T(\mu)$ such that for all $t > T(\mu)$, $q(t) \in U_\mu(\bar{q}) \cap B^{-1}(1)$ and $p(t) \in U_\mu(\bar{p}) \cap B^{-1}(1)$, or $p(t) \in U_\mu(\bar{p}) \cap B^{-1}(3)$ where $[p,\delta_1] \cap B^{-1}(1) = \emptyset$ implies that we have $B(p(t')) = 2$ and $B(q(t')) = 1$ for some $t' > t$.

Given $U_\mu(\bar{p})$ and $U_\mu(\bar{q})$, by lemma 9 we can find $\mu_1, \mu_2 < \mu$ and $T(\mu, \mu_1, \mu_2)$ so that for all $t > T(\mu, \mu_1, \mu_2)$, $p(t) \in U_{\mu_1}(\bar{p})$ and $q(t) \in U_{\mu_2}(\bar{q})$ implies player 1's belief will leave $U_{\mu_1}(\bar{p})$ before player 2's leaves $U_{\mu_2}(\bar{q})$. Let $\{\varepsilon_n\}$ be a decreasing sequence of positive real number satisfying the following conditions:

(a) $\varepsilon_n < \mu_1$, $\varepsilon_n \rightarrow 0$;

(b) $p \in U_{\varepsilon_n}(\bar{p})$ implies $[p, \delta_1] \cap B^{-1}(1) \subseteq U_{\varepsilon_{n-1}}(\bar{p})$ and $[p, \delta_3] \cap B^{-1}(1) \subseteq U_{\varepsilon_{n-1}}(\bar{p})$.

Furthermore, choose $\delta < \mu_2$ such that $q \in U_\delta(\bar{q})$ implies $[q, \delta_1] \cap U_{\varepsilon_1}(\bar{p}) = \varnothing$ (see figure 10). Construct two subsequences of beliefs $\{p(\bar{t})\}$ and $\{q(\bar{t})\}$ as follows. Consider any $t_{n'} > \max\{T(\mu), T(\mu, \mu_1, \mu_2)\}$ so that $p(t_{n'}) \in U_{\varepsilon_2}(\bar{p})$ and $q(t_{n'}) \in U_\delta(\bar{q})$ (that such a $t_{n'}$ exists follows from the fact that $p(t_n) \rightarrow \bar{p}$ and $q(t_n) \rightarrow \bar{q}$). Now if $B(p(t_{n'})) = 1$ and $B(q(t_{n'})) = 1$ then it follows from above that we would have $B(p(t'')) = 2$ and $B(q(t'')) = 1$ for some $t'' > t_{n'}$, which is a combination we have ruled out. Therefore we have either $B(p(t_{n'})) = 1$ and $B(q(t_{n'})) = 3$, or $B(p(t_{n'})) = 3$ and $B(q(t_{n'})) = 1$, or $B(p(t_{n'})) = 3$ and $B(q(t_{n'})) = 3$ (if $\bar{p} = \delta_3$ only the first two combinations are possible).

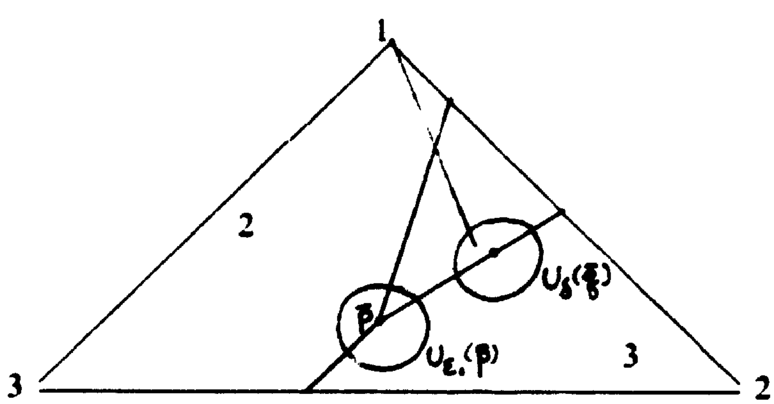

Figure 10

It follows from the properties of $U_{\varepsilon_2}(\bar{p})$ and $U_\delta(\bar{q})$ that $B(p(\bar{t}_1)) =$

$B(q(\bar{t}_1)) = 1$ for some $\bar{t}_1 > t_{n''}$. Note that $p(\bar{t}_1) \in U_{\varepsilon_1}(\bar{p})$ and $q(\bar{t}_1) \in U_{\varepsilon_1}(\bar{p})$. Now consider $t_{n''} > \bar{t}_1$ such that $p(t_{n''}) \in U_{\varepsilon_2}(\bar{p})$ and $q(t_{n''}) \in U_\delta(\bar{q})$. Again we can find a $\bar{t}_2 > t_{n''}$ such that $B(p(\bar{t}_2)) = B(q(\bar{t}_2)) = 1$, $p(\bar{t}_2) \in U_{\varepsilon_2}(\bar{p})$ and $q(\bar{t}_2) \in U_{\varepsilon_1}(\bar{p})$. By repeating this procedure we can construct two sequences $\{p(\bar{t}_r)\}$ and $\{q(\bar{t}_r)\}$ such that (1) $B(p(\bar{t}_r)) = B(q(\bar{t}_r)) = 1$; (2) $p(\bar{t}_r) \to \bar{p}$ and $q(\bar{t}_r) \in B^{-1}(1) \setminus U_{\varepsilon_1}(\bar{p})$ for all $\bar{t}_r$.

Without loss of generality let $q(\bar{t}_r) \to q'$. Obviously $q' \neq \bar{p}$. Like $\bar{p}$ and $\bar{q}$, we can find a $\mu' > 0$ such that $\tau(p, \delta_1) + \mu' < \tau(q, \delta_1)$ for all $p \in U_{\mu'}(\bar{p}) \cap B^{-1}(1)$ and all $q \in U_{\mu'}(q') \cap B^{-1}(1)$. Applying lemma 7 again we can find a $T(\mu')$ such that for all $t > T(\mu')$, $p(t) \in U_{\mu'}(\bar{p}) \cap B^{-1}(1)$ and $q(t) \in U_{\mu'}(q') \cap B^{-1}(1)$ implies that $B(p(t')) = 2$ and $B(q(t')) = 1$ for some $t' > t$. Now by our construction of $\{p(\bar{t}_r)\}$ and $\{q(\bar{t}_r)\}$ we have $p(\bar{t}_r) \in U_{\mu'}(\bar{p}) \cap B^{-1}(1)$ and $q(\bar{t}_r) \in U_{\mu'}(q') \cap B^{-1}(1)$ for all $\bar{t}_r$ large enough, which in turn implies that $B(p(t'')) = 2$ and $B(q(t'')) = 1$ for some very large t''. This results in a contradiction. Using similar arguments, one can show that $\bar{q}$ cannot be belong to other best-response regions of $S_1 \setminus \{\bar{p}\}$.

Therefore the only pair of limit points involving $\bar{p}$ is $(\bar{p}, \bar{p})$. We now assert that $(\{p(t)\}, \{q(t)\})$ converges to $(\bar{p}, \bar{p})$. Suppose the contrary. Then there exist $\varepsilon, \xi > 0$ such that $p(t) \in S_1 \setminus U_\varepsilon(\bar{p})$ and $q(t) \in S_1 \setminus U_\xi(\bar{p})$ for an infinite number of ts. Now it is not difficult (although tedious) to show that for some $\varepsilon' < \min\{\varepsilon, \xi\}$ and $T_{\varepsilon'}$ sufficiently large, $p(t') \in U_{\varepsilon'}(\bar{p})$ and $q(t') \in U_{\varepsilon'}(\bar{p})$ for some $t' > T_\varepsilon$ implies that $p(t) \in U_\varepsilon(\bar{p})$ and $q(t) \in U_\varepsilon(\bar{p})$ for all $t > t'$. Since we know that $(\bar{p}, \bar{p})$ is a limit point of $(\{p(t)\}, q\{(t)\})$ we can be sure that both players' beliefs will fall into the neighbourhood $U_{\varepsilon'}(\bar{p})$ for some $t' > T_{\varepsilon'}$ and hence $p(t), q(t) \in U_{\varepsilon}(\bar{p})$ for all $t' > T_{\varepsilon'}$. This contradiction proves our assertion.

Now consider case (ii). Two additional possibilities arise for this case: $\bar{p} \in S_1 \setminus (\delta_1, \delta_2)$ and $\bar{p} \in (\delta_1, \delta_2)$. When $\bar{p} \in S_1 \setminus (\delta_1, \delta_2)$ the strategy of the proof is the same as above and is omitted. When $\bar{p} \in (\delta_1, \delta_2)$ beliefs do not necessary converge to $(\bar{p}, \bar{p})$. In this case it can be shown that if $(\bar{p}, \bar{q})$ is not a Nash equilibrium then it cannot be a limit point of $(\{p(t)\}, \{q(t)\})$. On the other hand, since there is a continuum of equilibria involving $\bar{p}$ we may have more than one limit point for the sequence $(\{p(t)\}, \{q(t)\})$.

Nevertheless, one can still show that

$$\liminf_{t} \inf_{q} \left\{ \|p(t) - q\| \mid q \in \partial(1, 2) \right\} = \lim_{t} \inf_{q} \left\{ \|q(t) - q\| \mid q \in \partial(1, 2) \right\}$$
$$= 0$$

In other words, as t goes to infinity the pair of beliefs $(p(t), q(t))$ looks more and more like a mixed-strategy Nash equilibrium, although it may not converge to any particular equilibrium.

This completes our investigation of category (A). We now proceed to category (B), the class of 3×3 symmetric games that satisfy strategic complementarity. Again we find it convenient to distinguish between the two different subcases:

(B1) $B(\delta_1)$, $B(\delta_2)$ and $B(\delta_3)$ are all distinct;

(B2) $B(\delta_1)$, $B(\delta_2)$ and $B(\delta_3)$ are not all distinct.

Consider (B1) first. Obviously we must have $B(\delta_1) = 1$, $B(\delta_2) = 2$ and $B(\delta_3) = 3$. Also, strategic complementarity implies that $BR([\delta_1,$

$\delta_2]) = \{1, 2\}$ and $BR([\delta_2, \delta_3]) = \{2, 3\}$. If $B(p(t')) = B(q(t'))$ for some $t' \geq 1$ then by lemma 6 beliefs will converge to one of the three symmetric, pure-strategy equilibria. So suppose $B(p(t)) \neq B(q(t))$ for all $t \geq 1$. If $p(1) = \delta_1$ and $q(1) = \delta_2$, or $p(1) = \delta_2$ and $q(1) = \delta_1$, or $p(1) = \delta_2$ and $q(1) = \delta_3$, or $p(1) = \delta_3$ and $q(1) = \delta_2$, then we have $p(t) \in [\delta_1, \delta_2]$ and $q(t) \in [\delta_1, \delta_2]$, or $p(t) \in [\delta_2, \delta_3]$ and $q(t) \in [\delta_2, \delta_3]$ for all $t \geq 1$. By lemma 5 beliefs will converge to a Nash equilibrium.

If $p(1) = \delta_1$ and $q(1) = \delta_3$, or $p(1) = \delta_3$ and $q(1) = \delta_3$ there are two possible outcomes. If $B([\delta_1, \delta_3]) = \{1, 3\}$ then beliefs will always stay on the interval $[\delta_1, \delta_3]$. If $B([\delta_1, \delta_3]) = \{1, 2, 3\}$ (i.e. there exist $p_1, p_2 \in [\delta_1, \delta_2]$, where $p_1 \neq p_2$, such that $B(p) = 2$ for any $p \in [p_1, p_2]$) then by lemmas 1 and 2 $[p, \delta_2] \cap B^{-1}(1) = \emptyset$ for any $p \in B^{-1}(2)$, and $[p, \delta_3] \cap B^{-1}(1) = \emptyset$ for any $p \in B^{-1}(3)$. In this case we have $p(t') \in B^{-1}(2)$ and $q(t') \in B^{-1}(3)$ for some $t' > 1$, and the two players will alternate between strategies 2 and 3 for all remaining periods. Again by lemma 6 beliefs will converge.

Next consider (B2). If either $B^{-1}(1)$ or $B^{-1}(2)$ or $B^{-1}(3)$ is empty then it follows by lemma 5 that beliefs will converge. So suppose none of them is empty. Since $B(\delta_1)$, $B(\delta_2)$ and $B(\delta_3)$ are not all distinct we have, in view of strategic complementarity, $B(\delta_1) = B(\delta_2) = 1$ or $B(\delta_2) = B(\delta_3) = 3$. Both cases are qualitatively the same and it suffices to discuss only one of them, say, $B(\delta_2) = B(\delta_3) = 3$.

First note that if $B(p(t')) = B(q(t')) = 1$ or for some $t' \geq 1$ then $p(t)$, $q(t)$ will converge to (δ_1, δ_1) or (δ_3, δ_3). So assume that none of these two configurations ever occur. We further distinguish between the following two possibilities:

(B2.1) There does not exist a $\bar{p}$ such that $BR(\bar{p}) = \{1, 2, 3\}$;

(B2.2) There exists a $\bar{p}$ such that $BR(\bar{p}) = \{1, 2, 3\}$.

We will consider (B2.2) first because it is simpler. As in the analysing (A2.1) we apply lemma 3 and find a T large enough so that for all $t > T$, there will be no jump in either player's belief from the region $B^{-1}(1)$ to region $B^{-1}(3)$, or from $B^{-1}(3)$ to $B^{-1}(1)$. Also, in this case $B(p(t')) = B(p(t')) = 2$, or $B(p(t')) = 2$ and $B(q(t')) = 3$, or $B(p(t')) = 3$ and $B(q(t')) = 2$ for some $t' \geq 1$ implies that $B(p(t'')) = B(q(t'')) = 3$ or some $t'' > t'$. Therefore we can eliminate these combinations from our list as well.

Now the only possible combination of responses are $B(p(t)) = 1$ and $B(q(t)) = 2$; $B(p(t)) = 2$ and $B(q(t)) = 1$; $B(p(t)) = 1$ and $B(q(t)) = 3$; and $B(p(t)) = 3$ and $B(q(t)) = 1$. Consider any $t' > T$. If $B(p(t')) = 1$ and $B(q(t')) = 2$ (or $B(p(t')) = 2$ and $B(q(t')) = 1$) then it is easy to show that the two players will always change their strategies simultaneously and they alternate between 1 and 2. By lemma 5 beliefs will therefore converge. If $B(p(t')) = 1$ and $B(q(t')) = 3$ (or $B(p(t')) = 3$ and $B(q(t')) = 1$) then for some $t'' > t'$ we will have $B(p(t'')) = 1$ and $B(q(t'')) = 2$. Again beliefs will converge.

We now move to (B2.2) where there is a completely-mixed strategy equilibrium. It is useful to differentiate between the following two possbilities:

(1) There is a subsequence $\{p(t_m)\}$ or $\{q(t_m)\}$ such that $p(t_m) \longrightarrow \bar{p}$ or $q(t_m) \longrightarrow p$;

(2) No such subsequence exists.

Consider possibility (2) first. Since no subsequence of beliefs converge to $\bar{p}$, we can find an $\varepsilon > 0$ and a $\hat{T}$ such that $p(t) \in U_{\varepsilon}(\bar{p})$ and $q(t) \in U_{\varepsilon}(\bar{p})$ for all $t > \hat{T}$. As in the analysis of similar situation for (A2.2) we apply lemma 4 and find a $T(\varepsilon)$ such that $t > T(\varepsilon)$ and $[p(t), \delta_{B(q(t))}) \cap B^{-1}(2) \setminus U_{\varepsilon}(\bar{p}) \neq \emptyset$ or $[q(t), \delta_{B(p(t))}) \cap B^{-1}(2) \setminus U_{\varepsilon}(\bar{p}) \neq \emptyset$ implies that $B(p(t')) = 2$ or $B(q(t')) = 2$ for some $t' > t$. We also distinguish between two different situations:

(i) $\exists\ p_1, p_2 \in [\delta_1, \delta_2] \cap B^{-1}(2)$ where $p_1 \neq p_2$;

(ii) $\exists\ q_1, q_2 \in [\delta_1, \delta_3] \cap B^{-1}(2)$ where $q_1 \neq q_2$.

The outcome of situation (i) is exactly the same as (B2.1) and its proof is omitted to avoid repitition. For (ii), we first note that if $B(p(t')) = 2$ and $B(q(t')) = 1$ where $[q(t'), \delta_2) \cap B^{-1}(2) \setminus U_{\varepsilon}(\bar{p}) = \emptyset$ (or $B(q(t')) = 2$ and $B(p(t')) = 1$ where $[p(t'), \delta_2) \cap B^{-1}(2) \setminus U_{\varepsilon}(\bar{p}) = \emptyset$) then player 1 will only use strategies 1 and 2 and player 2 will only use strategies 1 and 3 (or player 1 uses only strategies 1 and 3 and player 2 uses only 1 and 2) for all $t > t'$ and they always switch their strategies simultaneously. Now using the same method of reasoning as we have done in the corresponding situation in (A2.2) it can shown that we must have $B(p(t')) = 2$ and $B(q(t')) = 1$ where $[q(t'), \delta_2) \cap B^{-1}(2) \setminus U_{\varepsilon}(\bar{p}) = \emptyset$, or $B(q(t')) = 2$ and $B(p(t')) = 1$ where $[p(t'), \delta_2) \cap B^{-1}(2) \setminus U_{\varepsilon}(\bar{p}) = \emptyset$ for some $t' > \max\{\hat{T}, T(\varepsilon)\}$. Hence by lemma 5 beliefs will converge to some mixed-strategy equilibrium. We now return to possibility (1) above where there is a subsequence of beliefs converging to $\bar{p}$. Like (2) we need to distinguish between two different situations: (i) $\exists\ p_1, p_2 \in [\delta_1, \delta_2] \cap B^{-1}(2)$ where $p_1 \neq p_2$; (ii) $\exists\ q_1, q_2 \in [\delta_1, \delta_3] \cap B^{-1}(2)$ where $q_1 \neq q_2$.

Consider (i) first. We first show that for any $\bar{q} \neq \bar{p}$ the pair $(\bar{p},$

Proof of lemma 11. We first show that if a limit cycle exists then (i) for all $p \in B^{-1}(i)$, $i = 1, 2, 3$, $[p, \delta_i] \cap B^{-1}(j) = \emptyset$ whenever $j \neq i$ and $j \neq B(\delta_i)$; (ii) $i \notin BR(\delta_i)$ for $i = 1, 2, 3$; (iii) $k \notin BR(\delta_i)$ for all $k \neq B(\delta_i)$.

To show (i) suppose $[\bar{p}, \delta_i] \cap B^{-1}(j) \neq \emptyset$ for some $\bar{p} \in B^{-1}(i)$ and $j \neq i$, $j \neq \delta_i$. Note that j must be strictly better than i against δ_i or otherwise we have $[\bar{p}, \delta_i] \cap B^{-1}(j) = \emptyset$. By lemma 10. there exists an interval $[a_M, a_{M+1}]$, $1 \leq M \leq N$ such that $B((a_M, a_{M+1})) = \{ i \}$ and $B((a_{M-1}, a_M)) \neq \{ i \}$. We have either $a_M \in \partial(i, B(\delta_i))$ or $a_M \in \partial(i, j) \backslash \partial(i, B(\delta_i))$. Since $[a_M, a_{M+1}] \subseteq [a_M, \delta_i]$ the first case is impossible by convexity of $B^{-1}(B(\delta_i))$. So we must have $a_M \in \partial(i, j) \backslash \partial(i, B(\delta_i))$. But since j is a strictly better response than i against δ_i j must be strictly better than i against any $p \in [a_M, \delta_i]$. This implies that $B((a_M, a_{M+1})) = \{ j \}$, which is a contradiction.

For (ii) note that we cannot have $3 \in BR(\delta_3)$ since $B(\delta_3) \neq 3$ by hypothesis. So suppose $i \in BR(\delta_i)$ for some $i \in \{1, 2\}$. By lemma 10 there exists $[a_M, a_{M+1}] \subseteq C$ such that $B((a_M, a_{M+1})) = \{ i \}$, $[a_M, a_{M+1}] \subseteq [a_M, \delta_i]$ and $B((\delta_{M+1}, \delta_{M+2})) \neq \{ i \}$. Now by convexity of $B^{-1}(i)$ we have $B(a_M) = i$. If $i = 1$ then we must have $BR(a_M) = \{ 1 \}$, which implies that $B(a_{M+1}, a_{M+2})) = \{ 1 \}$, a contradiction. Therefore $i = 2$ and hence $B((a_{M+1}, a_{M+2})) = \{ 1 \}$. Since $[a_{M+1}, a_{M+2}] \subseteq [a_{M+1}, \delta_1]$ by convexity of $B^{-1}(2)$ we have $B(\delta_1) = 3$. Let $[a_{M'}, a_{M'+1}] \subseteq C$ be such that $B((a_m, a_{m+1})) = \{ 1 \}$ for $M+1 \leq m < M'$ and $B((a_{M'}, a_{M'+1})) = \{ 3 \}$ (the existence of $a_{M'}$ follows from (i)). Since $BR(a_m) = \{1, 3\}$ by convexity of $B^{-1}(1)$ we have $B(\delta_3) = 2$. This implies $BR(\delta_2) = \{ 1 \}$, which contradicts our hypothesis.

To show (iii) suppose $k \in BR(\delta_i)$ for some $k \notin B(\delta_i)$. From (ii) we cannot have $k = i$. Let $B(\delta_i) = j$ and hence $j > k$. Since $B(\delta_1)$, $B(\delta_2)$ and $B(\delta_3)$ are all distinct this must imply $B(\delta_j) = k$ and $B(\delta_k) = i$. By convexity of $B^{-1}(j)$ we have $B((\delta_i, \delta_j)) = \{ k \}$. Now it follows from (i) that there exists an $\varepsilon > 0$ such that $B(p) = k$ for all $p \in B_\varepsilon(\delta_i) \cap S^3_{1++}$, where S^3_{1++} denotes the interior of S^3_1. Consider any $p' \in B^{-1}(i) \cap S^3_{1++}$ (that $B^{-1}(i) \cap S^3_{1++}$ is nonempty follows from (ii)). We have $[p', \delta_i] \cap B^{-1}(k) \neq \varnothing$, which contradicts (i).

(i), (ii) and (iii) together imply $BR(\delta_i) = \{ B(\delta_i) \}$, $i = 1, 2, 3$. For convenience reference we label the pure strategies so that $B(\delta_1) = 2$, $B(\delta_2) = 3$ and $B(\delta_3) = 1$. It follows from (i) that $\partial(1, 2)$, $\partial(2, 3)$ and $\partial(3, 1)$ are nonempty and that $[p, \delta_i] \cap \partial(i, B(\delta_i))$ is a singleton for all $p \in B^{-1}(i)$, $i = 1, 2, 3$. Consider any $p^1 \in B^{-1}(2)$. Let $f(p^1)$, $g(f(p^1))$ and $h(g(f(p^1)) = p^2$ be the unique elements of $\partial(2, 3) \cap [p^1, \delta_2]$, $\partial(3,1) \cap [f(p^1), \delta_3]$ and $\partial(1,2) \cap [g(f(p^1)), \delta_1]$ respectively. Define $f(p^2)$, $g(f(p^2))$ and p^3 in a similar way. Repeating this process indefinitely we obtain a sequence $\{p^n\}$. It can be easily seen geometrically (see figure 2) that if $p^2[1] > p^3[1]$ (resp. $p^2[1] < p^3[1]$) then $p^n[1] > p^{n+1}[1]$ (resp. $p^n[1] < p^{n+1}[1]$) for all $n \geq 2$. Therefore p^1 lies on a limit cycle only if $p^2[1] = p^3[1]$. On the other hand, if $p^2[1] = p^3[1]$ then it is straightforward to verify that $[p^2, f(p^2)] \cup [f(p^2), g(f(p^2))] \cup [g(f(p^2)), p^2]$ defines a limit cycle. Consequently if p^1 lies on a limit cycle then it must be the case that $p^1 \in [p^2, f(p^2)]$.

Now suppose $C = [\bar{p}, f(\bar{p})] \cup [f(\bar{p}), g(f(\bar{p}))] \cup [g(f(\bar{p})), \bar{p}]$ is a limit cycle. We need to show that for any ε-neighbourhood containing C,

there exists a $T(\varepsilon)$ such that $p(t) \in B_\varepsilon(C)$ for all $t > T$. First we notice that for any $p^1 \in B^{-1}(2)$ if $p^1[1]/p^1[3] > \bar{p}[1]/\bar{p}[3]$ (resp. $p^1[1]/p^1[3] < \bar{p}[1]/\bar{p}[3]$) then $p^1[1] > p^2[1]$ (resp. $p^1[1] < p^2[1]$) and $p^1[1]/p^1[3] > p^2[1]/p^2[3] > \bar{p}[1]/\bar{p}[3]$ (resp. $p^1[1]/p^1[3] < p^2[1]/p^2[3]$
$< \bar{p}[1]/\bar{p}[3]$.

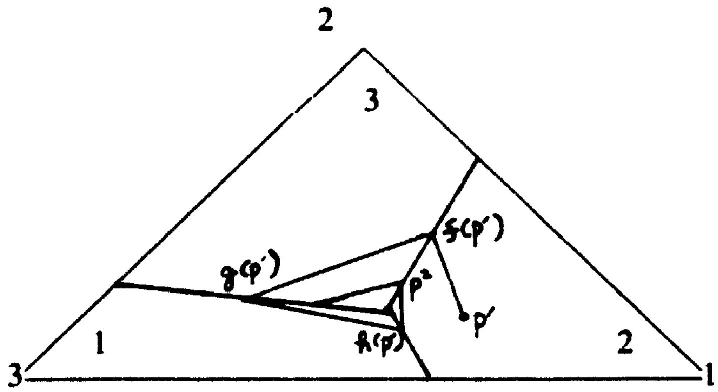

Figure 2.

For any $p^1 \in B^{-1}(2)$ and $\delta \geq 0$ we define

$$\psi(p^1, \delta) = \Big\{p' \in \partial(1,2) \mid \|p' - h(b)\| < \delta,\ \|b - g(q)\| < \delta \text{ and } \|q - f(p^1)\| < \delta \Big\}$$

$$\phi(p^1, \delta) = \mathrm{Inf}\Big\{p^1[1]/p^1[3] - p'[1]/p'[3] \mid p' \in \psi(p^1, \delta)\Big\}$$

$$G(\varepsilon, 0) = \Big\{p \in B^{-1}(2)\backslash B_\varepsilon(C) \mid p[1]/p[3] - \bar{p}[1]/\bar{p}[3] > 0\Big\}$$

$$G(\varepsilon, 1) = \Big\{p \in B^{-1}(2)\backslash B_\varepsilon(C) \mid p[1]/p[3] - \bar{p}[1]/\bar{p}[3] < 0\Big\}$$

$$\varphi(\varepsilon, \delta, 0) = \mathrm{Inf}\Big\{\phi(p^1, \delta) \mid p^1 \in G(\varepsilon, 0)\Big\} \text{ and}$$

$$\varphi(\varepsilon, \delta, 1) = \mathrm{Inf}\Big\{\phi(p^1, \delta) \mid p^1 \in G(\varepsilon, 1)\Big\}$$

By Theorem of maximum both $\varphi(\varepsilon, \delta, 0)$ and $\varphi(\varepsilon, \delta, 1)$ are continuous

functions of (ε, δ). In particular $\lim_{\delta \to \infty} \varphi(\varepsilon, \delta, 0) > 0$ and $\lim_{\delta \to \infty} \varphi(\varepsilon, \delta, 1) < 0$ for all $\varepsilon > 0$. Therefore for every $\varepsilon > 0$ we can find a $\delta(\varepsilon) > 0$ such that $\varphi(\varepsilon, \delta', 0) > 0$ and $\varphi(\varepsilon, \delta', 1) < 0$ for all
$\delta' \leq \delta(\varepsilon)$.

Consider any sequence $\{p(t)\}$ that satisfies (3). If $p(t) \notin B^{-1}(2)$ for all t large enough then by lemma 5 $(p(t), p(t))$ must converge to a Nash equilibrium. But the only equilibrium of this game is the completely-mixed strategy equilibrium. This results in a contradiction. Thus $p(t) \in B^{-1}(2)$ for infinitely many ts. For any $\varepsilon > 0$ choose any $\varepsilon' \in (0, \varepsilon)$ and T so that $\frac{1}{T} < \min\{\delta(\varepsilon'), \varepsilon - \varepsilon'\}$. Suppose $p(t') \in B^{-1}(2)$ and $p(t')[1]/p(t')[3] > \bar{p}[1]/\bar{p}[3]$ for some $t' \geq T$. Let $\{t'_n\}$ and $\{t''_n)\}$, $t'_{n+1} > t''_n \geq t'_n > t'$, be two subsequences of times satisfiying $p(t) \in B^{-1}(2)$ for $t'_n \leq t \leq t''_n$ and $p(t) \notin B^{-1}(2$ for $t''_n \leq t \leq t'_{n+1}$. By virtue of our construction we have:

$$p(t'_n)[1]/p(t'_n)[3] > p(t'_{n+1})[1]/p(t'_{n+1})[3] > \bar{p}[1]/\bar{p}[3]$$

for all $p(t'_n) \notin B_{\varepsilon'}(C)$. That is, $p(t)$ will get closer to the neighbourhood $B_{\varepsilon'}(C)$ every time it enters the region $B^{-1}(2)$ until it eventually enters the neighbourhood. Since

$$\lim_{n \to \infty} \| p(t'_{n+1}) - h(g(f(p(t''_n)))) \| = 0$$

it follows from the above discussion that $p(t) \in B_{\varepsilon'}(C)$, $t'_n \leq t \leq t''_n$, for all t'_n large enough. Moreover, once inside the neighbourhood $p(t)$ will not get out of the bigger open set $B_{\varepsilon}(C)$. Similar result is obtained if $p(t)[1]/p(t)[3] < \bar{p}[1]/\bar{p}[3]$ for all $p(t) \in B^{-1}(2)$.

By repeating the previous argument for the other two regions $B^{-1}(1)$ and $B^{-1}(3)$ we can show that $d(p(t), B_{\varepsilon'}(C)) = 0$ for all t large enough.

As ε is arbitrary this completes our proof.

Q.E.D.

Proof of lemma 12. We distinguish between the following three cases:

(1) $i \in BR(\delta_i)$ for some $i \in \{1, 2, 3\}$;

(2) $BR(\delta_i)$ is a singleton and $[p, \delta_i] \cap B^{-1}(j) = \emptyset$ for all $i \in \{1, 2, 3\}$, $j \neq i$ and $j \neq B(\delta_i)$;

(3) $[p, \delta_i] \cap B^{-1}(j) \neq \emptyset$ for some $i \in \{1, 2, 3\}$, $j \neq i$ and $j \neq B(\delta_i)$;

Consider (i) first. If there is a t' such that $B(p(t)) = i$ then $p(t)$ will converge to δ_i by lemma 6. Otherwise $(p(t), q(t))$ converges to a mixed-strategy equilibrium by lemma 5.

Next consider case (ii). It happens that we can apply the same argument as we have used in the proof of lemma 11 to show that $(p(t), q(t))$ must converge to the unique completely-mixed strategy of the game.

Finally consider case (iii). For convenient reference we label the pure strategies so that $B(\delta_1) = 2$, $B(\delta_2) = 3$ and $B(\delta_3) = 1$. Without loss of generality suppose $[p, \delta_1] \cap B^{-1}(3) \neq \emptyset$ for some $\bar{p} \in B^{-1}(1)$. There are two possibilities:

(a) there does not exist a $\bar{p} \in S_1^3$ with $BR(\bar{p}) = \{1, 2, 3\}$;

(b) there exists a $\bar{p} \in S_1^3$ with $BR(\bar{p}) = \{1, 2, 3\}$;

In the case of possibility (a), we can, as we have done in the proof of Theorem 1, apply lemma 3 and lemma 5 to show that $(p(t), q(t))$ must converge to some mixed-strategy equilibrium.

For possibility (b), we need to differentiate between the following two subcases:

(b1) there does not exist a subsequence $\{p(t_k)\}$ converging to $\bar{p}$;

(b2) there exists a subsequence $\{p(t_k)\}$ converging to $\bar{p}$.

Suppose we have (b1). In this case we can find an $\varepsilon > 0$ such that $p(t) \notin B_\varepsilon(\bar{p})$ for all t large enough. By our hypothesis we can also find some $\varepsilon_1 > 0$ such that $[p, \delta_3] \cap B^{-1}(2) = \emptyset$ for any $p \in B_{\varepsilon_1}(p')$ and $p' \in \partial(1,3)\backslash B_\varepsilon(\bar{p})$, and $[q, \delta_1] \cap B^{-1}(3) \neq \emptyset$ for any $q \in B_{\varepsilon_1}(q')$ and $q' \in \partial(1,3)\backslash B_\varepsilon(\bar{p})$. Next we apply lemma 4 and find a $T(\varepsilon)$ so that for all $t > T(\varepsilon)$, $[p(t), \delta_1] \cap B^{-1}(3)\backslash B_\varepsilon(\bar{p}) \neq \emptyset$, $p(t) \in B^{-1}(1)$.

Since $B(p(t)) = 1$ for infinitely many ts (otherwise $(p(t), q(t))$ will converge to some mixed-strategy equilibrium, which is impossible since $(\bar{p}, \bar{p})$ is the only equilibrium) there exists some $t' > T(\varepsilon)$ such that (i) $p(t') \in B^{-1}(1) \cap B_{\varepsilon_1}(q')$ for some $q' \in \partial(1,3)\backslash B_\varepsilon(\bar{p})$, and for all $t > t'$ (ii) $p(t-1) \in B_{\varepsilon_1}(\partial(1, 3))\backslash B_\varepsilon(\bar{p})$, $[p(t-1), \delta_3] \cap B_\varepsilon(\bar{p}) = \emptyset$ and $[p(t-1), \delta_1] \cap B_{\varepsilon_1}(\partial(1,3))\backslash B_\varepsilon(\bar{p}) \neq \emptyset$ implies $p(t+1) \in B_{\varepsilon_1}(\partial(1, 3))$. Let $\{p(t_n)\}$ be a convergent subsequence such that $p(t_n-1) \in B_{\varepsilon_1}(\partial(1, 3))\backslash B_\varepsilon(\bar{p})$ and $p(t_n) \notin B_{\varepsilon_1}(\partial(1, 3))\backslash B_\varepsilon(\bar{p})$. By our construction we have either $[p(t_n-1), \delta_3] \cap B_\varepsilon(\bar{p}) \neq \emptyset$ or $[p(t_n-1), \delta_1] \cap B_{\varepsilon_1}(\partial(1,3))\backslash B_\varepsilon(\bar{p}) = \emptyset$ or both. This means that $p(t_n)$ must jump across the open set $B_\varepsilon(\bar{p})$. Since $\lim_n \|p(t_n) - p(t_{n-1})\| = 0$ the distance of the jump must converge to zero, which is impossible from a geometrical point of view.

Now consider (b2). We will show that for any $\varepsilon > 0$ there exists a $T(\varepsilon)$ such that for all $t > T(\varepsilon)$, $p(t) \in B_\varepsilon(\bar{p})$ implies $p(t+k) \in B_\varepsilon(\bar{p})$ for $k \geq 0$. Let $\varepsilon'' < \varepsilon' < \varepsilon$ where $\varepsilon' - \varepsilon'' < \varepsilon - \varepsilon'$ and choose $T(\varepsilon')$

according to lemma 4. As in b1 we find an $\varepsilon_1 < \varepsilon' - \varepsilon''$ such that some $\varepsilon_1 > 0$ such that $[p, \delta_2] \cap B^{-1}(2) = \emptyset$ for any $p \in B_{\varepsilon_1}(\partial(1,3)) \setminus B_{\varepsilon}(\bar{p})$, and $[q, \delta_1] \cap B^{-1}(3) \neq \emptyset$ for any $q \in B_{\varepsilon_1}(\partial(1,3)) \setminus B_{\varepsilon}(\bar{p})$. If $[\bar{\bar{p}}, \delta_2] \cap B^{-1}(1) \neq \emptyset$ for some $\bar{\bar{p}} \in B^{-1}(1)$ and/or $[\hat{p}, \delta_3] \cap B^{-1}(2) \neq \emptyset$ for some $\hat{p} \in B^{-1}(1)$ we can do the same thing and find corresponding neigbourhood/s $B_{\varepsilon_2}(\partial(1,2)) \setminus B_{\varepsilon''}(\bar{p})$ and/or $B_{\varepsilon_3}(\partial(2, 3)) \setminus B_{\varepsilon''}(\bar{p})$.

Finally we choose some $T' > \max\{T(\varepsilon), \frac{1}{\varepsilon' - \varepsilon''}\}$ with the property that for all $t > T'$, if $p(t) \in B_{\varepsilon_1}(\partial(1, 3)) \setminus B_{\varepsilon''}(\bar{p})$ we have $[p(t), \delta_2] \cap B_{\varepsilon''}(\bar{p}) = \emptyset$ and $[p(t), \delta_1] \cap B_{\varepsilon_1}(\partial(1, 3)) \setminus B_{\varepsilon''}(\bar{p}) \neq \emptyset$ implies $p(t+1) \in B_{\varepsilon_1}(\partial(1, 3)) \setminus B_{\varepsilon''}(\bar{p})$ (if $[\bar{\bar{p}}, \delta_2] \cap B^{-1}(1) \neq \emptyset$ for some $\bar{\bar{p}} \in B^{-1}(1)$ and/or $[\hat{p}, \delta_3] \cap B^{-1}(2) \neq \emptyset$ for some $\hat{p} \in B^{-1}(1)$ we can choose T' with the corresponding property for $B_{\varepsilon_2}(\partial(1, 2)) \setminus B_{\varepsilon''}(\bar{p})$ and/or $B_{\varepsilon_3}(\partial(2, 3)) \setminus B_{\varepsilon''}(\bar{p})$).

Now since there is a subsequence $\{p(t_k)\}$ converging to $\bar{p}$ there exists a $t > T'$ such that $p(t) \in B_{\varepsilon''}(\bar{p})$. By our construction we should have $p(t+k) \in B_{\varepsilon}(\bar{p})$ for all $k \geq 0$. In fact, if $p(t+r) \in B_{\varepsilon''}(\bar{p})$ and $p(t+r+1) \notin B_{\varepsilon''}(\bar{p})$ it must be the case that $p(t+r+1) \in B_{\varepsilon'}(\bar{p})$ and we must have $p(t+r+2)$, $p(t+r+3)$, ... moving back to $B_{\varepsilon''}(\bar{p})$ (while staying within $B_{\varepsilon}(\bar{p})$ until belief enters the neighbourhood $B_{\varepsilon''}(\bar{p})$ again.

This completes our proof of lemma 12.

Q.E.D.

BIBLIOGRAHY

CHAPTER 1

[1] Aumman, R. and A. Brandenburger. (1991): "Epistemic Conditions for Nash equilibrium ", Working Paper 91-042, Harvard Business School.

[2] Brandenburger A. (1992): "Knowledge and Equilibrium in Games", *Journal of Economic Perspective*, 6, 83-101.

[3] Brown, G.W. (1951): "Iterative Solutions of Games by Fictitious Play", in *Activity Analysis of Production and Allocation*: New York: Wiley.

[4] Fundenberg, D. and D.M. Kreps (1993): "Learning Mixed Equilibria", *Games and Economic Behavior*, 5, 320-367.

[5] Jordan, J. (1991): "Bayesian Learning in Normal Form Games", *Games and Economic Behavior*, 3, 60-81.

[6] Kalai, E. and E. Lehrer. (1993): "Rational Learning Leads to Nash Equilibrium", mimeo, Northwestern University.

[7] Krishna, V. (1992): "Learning in Games with Strategic Complementarities", mimeo, Harvard Business School.

[8] Mailath, G.J. (1992): "Introduction: Symposium on Evolutionary Game Theory", *Journal of Economic Theory*, 57, 259-277.

[9] Milgrom, P. and J. Roberts. (1990): "Rationalizability, Learning and Equilibrium in Games with Strategic Complementarities", *Econometrica*, 58, 1255-1277.

[10] Miyasawa, K. (1961): "On the Convergence of the Learning Process in a 2x2 Non-zero Sum Game", Econometric Research Program,

Research Memorandum No. 33, Princeton University, Princeton.

[11] Robinson, J. (1951): "An Iterative Method of Solving a Game", Annals of Mathematics, 54, 296-301.

[12] Shapley, L. (1964): "Some Topics in Two-Person Games", in *Advances in Game Theory, Annals of Mathematical Studies*, vol. 5, pp 1-28.

CHAPTER 2

[13] Arrow, K.J. and G. Debreu. (1954): "Existence of an Equilibriumfor a Competitive Economy," *Econometrica*, 22, 265-90.

[14] Arrow, K.J. and F. Hahn.: General Competitive Analysis, San Francisco: Holden-Day, 1971.

[15] Blanchard, O. and N. Kiyotaki. (1987): "Monopolistic Competition and Effects of Aggregate Demand," *American Economic Review*, 647-66.

[16] Cooper, R. and A. John. (1988): "Coordinating Coordination Failures in Keynesian Models," *Quarterly Journal of Economics*, CIII, 441-63.

[17] Debreu, G. (1970): "Economies with a Finite Number of Equilibria," *Econometrica*, 38(3), 387-92.

[18] Debreu, G.: "Theory of Value, Cowles Foundation, Yale University, 1959.

[19] Dubey, P. (1978): "Finiteness and Inefficiency of Nash Equilibria", Cowles Foundation Discussion Paper No.508R, Yale University.

[20] Dubey, P. (1980): "Nash Equilibria of Strategic Market Games: Finiteness and Inefficiency," *Journal of Economic Theory*, 22, 363-376.

[21] Hart. O. (1985): "Monopolistic Competition in the Spirit of Chamberlin: A General Model," *Review of Economic Studies*. 52. 529-46.

[22] Jones. L. (1984): " A Competitive Model of Commodity Differentiation". *Econometrica*. 52(2). 507-30.

[23] Mas-Colell. A.: "The Cournotian Foundation of Walrasian Equilibrium Theory: An Exposition of Recent Theory ." in Advances in Economic Theory. ed. W. Hildenbrand. 183-224. Cambridge. Cambridge University Press. 1982.

[25] Mas-Colell. A.: The Theory of General Economic Equilibria: A Differentiable Approach. Cambridge. Cambridge University Press. 1985.

[26] Novshek. W. and H. Sonnenschein. (1978): "Cournot and Walrasian Equilibria." 22(9). 223-66.

[27] Mclennan. A. and M. Feldman. (1990): "Incomplete Learning in a Repeated Statistical Problem," mimeo. University of Minnesota.

[28] Negishi. T. (1961): "Monopolistic Competition and General Equilibrium," *Review of Economic Studies*. 28. 196-201.